Algebra general exam

August 15, 2017

Your name:

- Please show all your work and justify any statements that you make.
- State any theorem you use clearly and fully.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement of an earlier question proven in order to solve a later one.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. (10 points) Denote by D_{2n} the dihedral group of order $2n$, $n = 1$ being admitted. For which natural numbers m and n is D_{4nm} isomorphic to the direct product $D_{2m} \times D_{2n}$?

2. (10 points) Let G be a group of order $16 \cdot 11 \cdot 13 \cdot 17$. Assume that G has a normal nonabelian Sylow 2-subgroup. Show that the center of G is nontrivial.
   Remark: The claim remains true if G has an abelian normal Sylow 2-subgroup but then it is a bit harder to prove.

3. (16 points) Let p be a prime, n a natural number and $A \in GL_n(\mathbb{F}_p)$ diagonalizable over the algebraic closure $\overline{\mathbb{F}_p}$.
   (a) (4 points) Show that the order of A in $GL_n(\mathbb{F}_p)$ is equal to the lcm of the orders of the eigenvalues of A in $\overline{\mathbb{F}_p}^{\times}$.
   (b) (8 points) Prove that $GL_n(\mathbb{F}_p)$ has an element of order $p^n - 1$ which is diagonalizable over $\overline{\mathbb{F}_p}$.
   (c) (4 points) Explicitly construct an element of order 8 of $GL_2(\mathbb{F}_3)$.

4. (12 points) Let R be a commutative ring with 1 which is *Artinian*, i.e. for any descending chain $I_1 \supseteq I_2 \ldots \supseteq I_n \ldots$ of ideals of R there exists an n_0 such that $I_n = I_{n_0}$ for all $n \geq n_0$. Prove the following:
   (a) (4 points) If R is an integral domain, then it is a field.
   (b) (4 points) Any prime ideal of R is maximal.
   (c) (4 points) R has only finitely many maximal ideals.

5. (12 points) Let $R = \mathbb{Z}[x]/(x^3+x^2+1)$ be the quotient of the polynomial ring $\mathbb{Z}[x]$ modulo the principal ideal (x^3+x^2+1).
(a) (4 points) Is R an integral domain?
(b) (8 points) Which of the principal ideals $(2), (3), (5)$ of R are prime ideals? And which of them are maximal?

6. (15 points) Let $M|K, M|L, K|F, L|F$ be finite field extensions. Assume that for $\alpha, \beta \in M$, $K = F(\alpha)$, $L = F(\beta)$ and $M = F(\alpha, \beta)$. Set $a = [K : F]$ and $b = [L : F]$.
(a) (3 points) If $K|F$ and $L|F$ are Galois, show that also $M|F$ is Galois.
(b) (8 points) If $K|F$ and $L|F$ are Galois, prove that $[M : F]$ divides ab.
(c) (4 points) Give an example of M, K, L, F as above (but without the Galois assumption) such that $[M : F]$ does *not* divide ab.

7. (15 points) Consider the polynomial ring $R = F[x, y]$ in two variables over the field F and the ideal $I = (x, y)$ of R. Let $\phi : R \longrightarrow F$ be the F-algebra homomorphism with $\phi(x) = \phi(y) = 0$, which turns F into an R-module.
(a) (4 points) Show that the R-modules $F \otimes_R F$ and F are isomorphic.
(b) (2 points) Define maps $s, t : I \longrightarrow F$ by $s(f) = c_{1,0}$, respectively, $t(f) = c_{0,1}$ if $f = \sum_{i,j} c_{i,j} x^i y^j \in I$ (with $c_{i,j} \in F$ for all $i, j \in \mathbb{N}_0$). Verify that s and t are R-module homomorphisms.
(c) (6 points) Prove that $x \otimes y - y \otimes x$ is not 0 in $I \otimes_R I$.
(d) (3 points) Prove that I is not a flat R-module.

8. (10 points) Consider the $\mathbb{C}$-vector space $V = M_2(\mathbb{C})$ of complex 2×2 matrices and the linear transformation $T : V \longrightarrow V$ defined by $T(X) = AX - XA$ for all $X \in M_2(\mathbb{C})$, where A is the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Determine (as a 4×4 matrix) the Jordan canonical form of T.