Algebra general exam

January 12, 2017

Your name:

- Please show all your work and justify any statements that you make.
- State any theorem you use clearly and fully.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement of an earlier question proven in order to solve a later one.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Consider the polynomial $f(X) = X^4 - 2X^2 - 6$. Prove this polynomial is irreducible. Describe the splitting field of this polynomial (including its degree over $\mathbb{Q}$), and the Galois group of this splitting field (hint: pay attention to which roots are real and which are complex). (15pt)

2. Consider a field K, and two finite extensions L, M of K. Consider the K-algebra $L \otimes_K M$ (with the usual multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$). Prove that $L \otimes_K M$ is a field if and only if any time an extension E/K contains subfields L' and M' isomorphic to L and M, the composite LM has degree $[LM : K] = [L : K][M : K]$. (15pt)

3. Let R be a commutative ring, and M, N be R-modules. Show that for any submodules $M' \subset M$ and $N' \subset N$, the induced map $M \otimes_R N \to (M/M') \otimes_R (N/N')$ has kernel given by $M' \otimes N + M \otimes N'$ (hint: use the universal property of tensor products). (10pt)

4. We call a group G *polycyclic* if it contains a series of subgroups $\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$ such that G_i/G_{i-1} is a (possibly infinite) cyclic group.
   (a) Show that a finite group is polycyclic if and only if it is solvable. (5pt)
   (b) Show that $\mathbb{Q}$ is an example of an abelian group which is not polycyclic. (5pt)

5. Given a finite group G and two subgroups H, K, the *double cosets* of H and K are the sets of the form HgK for some $g \in G$.
   (a) Show that any two double cosets must be equal or disjoint. (5pt)
   (b) Show that the size of any double coset must divide the product of the orders $\#H \cdot \#K$. (5pt)
   (c) Find an example of a double coset whose size does not divide the order $\#G$. (5pt)

6. (a) Find the smallest integer n such that S_n has a subgroup of order 10, but S_k for $k < n$ does not. (5pt)
   (b) Find the smallest integer m such that S_m has an element of order 10, but S_k for $k < m$ does not. (5pt)

7. Consider the matrix

$$A = \begin{bmatrix} 12 & 4 & -16 \\ 4 & 3 & -7 \\ 8 & 3 & -11j \end{bmatrix}.$$

(a) Find the characteristic and minimal polynomials of this polynomial and its Jordan normal form. (8pt)

(b) Consider map $\mathbb{Z}^3 \to \mathbb{Z}^3$ induced by A. Describe the kernel and cokernel of this map as a sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. (7pt)

8. Let A be the ring of $n \times n$ matrices over a field F.

(a) Show the right ideals of A are precisely the subsets of the form

$$\{X \in A \mid \text{image}(X) \subset V\}$$

where V ranges over all linear subspaces of F^n. (5 pts)

(b) Show the left ideals of A are precisely the subsets of the form

$$\{X \in A \mid \text{kernel}(X) \supset W\}$$

where W ranges over all linear subspaces of F^n. (5 pts)

(c) Show that A is a simple ring: its only 2-sided ideals are A itself, and $\{0\}$. (5 pts)