Algebra general exam

August 17, 2016

- Please show all your work and justify any statements that you make.
- State any theorem you use clearly and fully.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement of an earlier question proven in order to solve a later one.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Consider the 3×3 matrix with entries in $\mathbb{Q}$

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & -3 \end{bmatrix}$$

   (a) Describe a field extension F of $\mathbb{Q}$ of minimal degree (either abstractly, or as a subfield of the complex numbers), such that A has an eigenvector with entries in F (note: you do **not** need to find the eigenvector or eigenvalue). (5 pts)
   (b) Determine if A is diagonalizable over $\mathbb{C}$. (5 pts)
   (c) Does there exist a 3×3 matrix with rational coefficients with no eigenvectors over $\mathbb{Q}$ which is not diagonalizable over $\mathbb{C}$? Find a counterexample, or prove none exists. (5 pts)

2. What is the smallest integer m such that there is a group of order m with no nontrivial normal p-subgroup for any prime p? (10 pts)

3. Let $f(x) = x^4 - x^2 + 1$.
   (a) Describe a splitting field E for f over $\mathbb{Q}$ (in particular, find its degree). (7 pts)
   (b) Describe the Galois group of E and all of its subfields. (8 pts)

4. Fix a group G.
   (a) Show that if N is a normal Sylow p-subgroup of G and H a subgroup of order not divisible by p, then HN is a subgroup of G isomorphic to a semi-direct product $N \rtimes H$. (5 pts)
   (b) Consider a group G of order 255. Show that G is cyclic. (10 pts)

5. Let V and W be vector spaces over a field F, and let $\{v_1, \dots, v_\ell\}$ and $\{w_1, \dots, w_\ell\}$ be elements of these vector spaces. Assume that the vectors $\{w_1, \dots, w_\ell\}$ are linearly independent. Show that $\sum_{i=1}^{\ell} v_i \otimes w_i = 0$ implies that $v_1 = \cdots = v_\ell = 0$. (10 pts)

6. (a) Name two examples of each of the following: (i) PIDs which are not fields (ii) UFDs which are not PIDs (iii) commutative integral domains which are not UFDs and (iv) commutative rings which are not integral domains, (v) noncommutative rings. (5 pts)
   (b) Given an example of a non-principal ideal in one of the examples you listed (with proof). (5 pts)

7. (a) Give a complete and irredundant list of abelian groups of order 144. (5 pts)
   (b) Give a complete and irredundant list of finitely generated modules over $\mathbb{F}_2[t]$ where the polynomial $t^4 + t^3 + t + 1$ acts trivially. (5 pts)

8. Let $n > 1$ and $m > 1$ be natural numbers, and c a complex number. Consider the associated $n \times n$ Jordan block

$$J_n(c) = \begin{bmatrix} c & 1 & 0 & \cdots & 0 \\ 0 & c & 1 & \cdots & 0 \\ 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c \end{bmatrix}$$

   (a) Show that $J_n(c)$ is a m-th power (i.e., there exists B such that $J_n(c) = B^m$) if and only if $c \neq 0$. (10 pts)
   (b) Show that any element of $GL_n(\mathbb{C})$ is an mth power. (5 pts)