ALGEBRA GENERAL EXAM January, 2016

You have four hours. Justify all your statements as much as possible, and show your work. State clearly any theorem you use. Do each problem on a separate sheet of paper and staple them together. You are to receive no help on this exam including from books, notes, internet, etc. Good luck.

1. (8 pts) Let K be a field (possibly finite). Prove that the polynomial ring $K[X]$ has infinitely many maximal ideals.

2. (8 pts) Let G be an infinite group and let H be a subgroup of finite index. Prove that there exists a subgroup K of H such that K has finite index in G and such that K is normal in G.

3. (8 pts) Let R be a commutative ring with identity. A non-zero R-module M is said to be *irreducible* if 0 and M are the only submodules of M. Prove that M is irreducible if and only if $M \cong R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R.

4. (10 pts) Let $G = S_n$ be the symmetric group on n elements, and let $\sigma = (123...n)$ be an n-cycle. Let K be the cyclic subgroup generated by σ. Prove that the order of the normalizer of K, i.e., the order of the subgroup $H = \{x \in S_n |\, x^{-1}\sigma x \in K\}$, is exactly $n \cdot \phi(n)$, where $\phi(n)$ is the Euler ϕ-function. (Recall that $\phi(n)$ is the number of positive integers less than n and relatively prime to n.)

5. (12 pts) Let T be a linear operator on a finite dimensional vector space V over a field F. Prove that

$$rank(T^3) + rank(T) \geq 2 \cdot rank(T^2).$$

6. (14 pts) Let $K = \overline{\mathbb{Q}}$ be the algebraic closure of the rationals in $\mathbb{C}$, i.e., the set of elements in the complex numbers which are algebraic over the rationals. By Zorn's lemma, there exists a maximal subfield of K, say E, which does not contain the square root of 2. Prove that every finite normal extension of E has cyclic Galois group. (Hint: reduce this to a question about groups.)

7. (16 pts) Let ϵ be a primitive 16th root of unity in the complex number. Set $s = \epsilon \cdot \sqrt{2}$. Let $E = \mathbb{Q}[\epsilon]$, where $\mathbb{Q}$ is the field of rational numbers, and set $f(X) = X^8 + 16 \in \mathbb{Q}[X]$. Show that that s is a root of $f(X)$. Prove that $\sqrt{2} \in \mathbb{Q}[\epsilon]$, and hence that $f(X)$ splits completely over E. If $G = Gal(E/\mathbb{Q})$, prove that no nonidentity element of G fixes s. Prove that $f(X)$ is irreducible over $\mathbb{Q}$.

8. (12 pts) Let K and L be finite extension fields of a field F of characteristic 0. Prove that $K \otimes_F L$ has no nonzero nilpotent elements. (Hint: use the primitive element theorem to represent K as a quotient of a polynomial ring over F.)

9. (12 pts) Let

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

Think of A as a matrix over the complex numbers. Find a 3 by 3 invertible matrix P such that $P^{-1}AP$ is in Jordan canonical form.