ALGEBRA GENERAL EXAM August, 2015

You have four hours. Justify all your statements as much as possible, and show your work. State clearly any theorem you use. Do each problem on a separate sheet of paper and staple them together. You are to receive no help on this exam including from books, notes, internet, etc. Put your name on the first page, and initials on all the other pages. Good luck.

1. (10 pts) Find all maximal ideals of $\mathbb{Z}[i]$ which contain 182. Find minimal generators for these ideals.

2. (10 pts) Let r_1, r_2, r_3 be the roots of the cubic polynomial X^3+10X^2-5X+4. Find the cubic polynomial with rational coefficients whose roots are r_1^2, r_2^2, r_3^2.

3. (20 pts)

   a) Let G be a group of order $2n$, where n is odd and $n > 1$. Prove that G cannot be simple. (Hint: consider elements of order 2 in the regular representation of G in S_{2n}.)

   b) Let $G = \mathbb{Z}_n^*$ denote the group of units in $\mathbb{Z}_n$. Find all integers n such that $x^2 = 1$ for all $x \in G$.

4 (15 pts). Find the characteristic polynomial, the minimal polynomial, and the Jordan canonical form of the matrix (over the complex numbers)

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

5 (10 pts). Let R be a commutative ring with identity, and let I be a nilpotent ideal, i.e., $I^k = 0$ for some k. Let M, N be two R-modules, and let $f : M \to N$ be an R-homomorphism. Suppose that the induced homomorphism from M/IM to N/IN is surjective. Prove that f is surjective.

6. (10 pts) Let F be a field and A an n by n matrix with coefficients in F. Assume that A has only one invariant factor. Prove that for every n by n matrix B with coefficients in F such that $AB = BA$ there is a polynomial $p(t) \in F[t]$ such that $p(A) = B$. (Hint: consider the structure of $V = F^n$ as a $k[A]$-module. Use that an endomorphism is determined by its action on a basis.)

7 (10 pts). Let V be a finite dimensional vector space over a field F and let V^* be its dual. For $v \in V$ and $f \in V^*$, denote by $\phi_{v,f}$ the endomorphism of V defined by $\phi_{v,f}(w) = f(w)v$ for $w \in V$. Prove that there exists a well-defined F-linear map $\Phi : V \otimes_F V^* \to \mathrm{End}_F(V)$ satisfying $\Phi(v \otimes f) = \phi_{v,f}$ for all $v \in V$ and $f \in V^*$. Prove that Φ is an isomorphism.

8 (15 pts). Consider the polynomial $x^6 - 3$ over the rational numbers. What is the degree of its splitting field, and what is the splitting field? Describe the Galois group of the splitting field as a subgroup of the symmetric group S_6. Is it abelian?