ALGEBRA GENERAL EXAM, AUGUST 18, 2014, 9AM–1PM

Directions.

- Show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will hurt your grade.
- Do each problem on a separate one-sided sheet of paper, and staple them together.
- The exam is pledged.

Throughout we denote $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Problem 1 (10 points). *Let G be a finite nilpotent group, $Z(G)$ its center and p a prime number. Prove that p divides $|G|$ if and only if p divides $|Z(G)|$.*

Problem 2 (9 points). *Compute*
(a) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$;
(b) $\mathbb{Z}_{2014} \otimes_{\mathbb{Z}} \mathbb{Z}_{2013}$;
(c) $Hom_{\mathbb{Z}}(\mathbb{Z}_{2014}, \mathbb{Z}_{10})$ *(Here $\mathbb{Z}_n$ is regarded as a $\mathbb{Z}$-module).*

Problem 3 (10 points). *Let p be a a prime number.*
(a) Determine the order of the automorphism group of $\mathbb{Z}_p \times \mathbb{Z}_p$;
(b) Prove that there exists a non-abelian group of order p^3.

Problem 4 (10 points). *Denote by J the $n \times n$ Jordan block with eigenvalue 0. For a positive integer k, determine the Jordan canonical form of J^k. (Hint: you can start by playing with some small values of k).*

Problem 5 (10 points). *Let K be a field and $a \in K$. Consider K as a $K[x]$-module (denoted by K_a) via the homomorphism $ev_a : K[x] \to K$ which is the identity on K and sends x to a. Let $K[[x]]$ be the power series ring, which is regarded as a $K[x]$-algebra in a natural way. Determine with proof the tensor product $K_a \otimes_{K[x]} K[[x]]$. (Hint: keep in mind if one needs to divide into cases depending on a.)*

Problem 6 (10 points). *(a) Compute the order of the group G of rigid motions of a regular octahedron O.*
(b) The group G acts on the set of vertices of O. Describe the stabilizer of a vertex of O.

Problem 7 (11 points). *Let $f(x) = x^5 - 2 \in \mathbb{Z}[x]$.*
(a) Determine the splitting field F of $f(x)$ over $\mathbb{Q}$;
(b) Determine the Galois group of $f(x)$ over $\mathbb{Q}$;
(c) List all the subfields K of F such that $[K : \mathbb{Q}] = 4$.

Problem 8 (10 points). *Let $A = M_n(R)$ be the algebra of $n \times n$ matrices over a commutative ring R with 1. Fix $1 \le i, j \le n$. Determine*
(a) the left ideal of A generated by E_{ij};
(b) the (two-sided) ideal of A generated by E_{ij};
(c) Are there possibly other nonzero ideals of A besides those of the form (b)?
(Here as usual E_{ij} denotes the matrix whose (i, j)th entry is 1 and 0 elsewhere.)