Algebra General Exam

August 19, 2013

Directions.

- Please show all your work and justify any statements that you make
- State clearly and fully any theorem you use
- Vague statements and hand-waving arguments will not be viewed favorable
- You may assume the statement for any early part of a problem in order to do a later part

Do each problem on a separate sheet of paper

(1) Let p be an odd prime and G a nonabelian group of order p^3.
  (a) (4 points) Prove that $|Z(G)| = p$
  (b) (4 points) Prove that $Z(G) = [G, G]$.

(2) (5 points) Let K and L be fields of characteristic 0. Prove that $K \bigotimes_{\mathbb{Z}} L$ is nonzero.

(3) If G is a group, then there is a natural action of Σ_n on $G^{\times n}$ given by permuting the factors. Define the wreath product $G \wr \Sigma_n$ to be

$$G \wr \Sigma_n = G^n \rtimes \Sigma_n$$

using this action of Σ_n on G^n.
  (a) (3 points) If X is a G-set, show that X^n is naturally a $G \wr \Sigma_n$ set by combining two actions: G^n on X^n via

$$(g_1, \dots, g_n) \cdot (x_1, \dots, x_n) = (g_1 x_1, \dots, g_n x_n)$$

  for $(g_1, \dots, g_n) \in G^n$ and $(x_1, \dots, x_n) \in X^n$, and
  Σ_n on X^n via

$$\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}),$$

  where $\sigma \in \Sigma_n$.
  (b) (3 points) Show that $\Sigma_n \wr \Sigma_m$ embeds into Σ_{nm}.
  (c) (3 points) Identify $\Sigma_2 \wr \Sigma_2$ with a more familiar group
  (d) (2 points) Determine the order of $G \wr \Sigma_n$ as a function of the orders of G and n
  (e) (2 points) Bonus: Determine (no proof needed) the p-Sylow subgroup of $\Sigma_{p^{k+1}}$ as a function of k. Provide no more than a sentence of justification.

(4) Let K be a field, and let $M_n(K)$ be the ring of $n \times n$ matrices with entries in K. For this problem, let $D \in M_n(K)$ be diagonalizable (over K) and, for each eigenvalue λ of D, let

$$E_\lambda := \{v \in K^n \mid Dv = \lambda v\}$$

be the corresponding eigenspace.

(a) (4 points) For any $A \in M_n(K)$, show that $AD = DA$ if and only if $A(E_\lambda) \subseteq E_\lambda$ for all eigenvalues λ of D.
(Hint: For the "if" part, you may use that $AD = DA$ if $ADv = DAv$ for all $v \in K^n$.)

(b) (4 points) If A is also diagonalizable and $AD = DA$, show that A and D are simultaneously diagonalizable (that is, there is a matrix P such that both PAP^{-1} and PDP^{-1} are diagonal). Provide a counter-example showing that this need not be the case if the matrices do not commute.

(c) (3 points) If D is invertible, show that the centralizer of D in $GL_n(K)$ is isomorphic to a direct product $GL_{n_1}(K) \times \ldots \times GL_{n_r}(K)$, where $n_1 + \ldots + n_r = n$. Also show that each of these products can be realized as the centralizer of some (appropriately chosen) D, provided that K has at least $n+1$ elements.

(5) Let F be a field and $f(x) = x^4 + 1 \in F[x]$.

(a) (3 points) Determine for which characteristic of F $f(x)$ is separable.

(b) (4 points) Assume that $f(x)$ is separable and irreducible over F, and denote by K the splitting field of $f(x)$ over F. Determine the Galois group $Gal(K|F)$.

(c) (4 points) If $f(x)$ is irreducible over F, prove first that F is infinite, and then that the characteristic of F is 0.

(6) Let p be a prime and ζ a primitive p^{th} root of unity (in $\mathbb{C}$). Set $R := \mathbb{Z}[\zeta]$ and $K := \mathbb{Q}(\zeta)$.

(a) (2 points) Show that R is a free $\mathbb{Z}$-module and $R \cap \mathbb{Q} = \mathbb{Z}$.

(b) (2 points) Identify $Gal(K|\mathbb{Q})$ and show that the natural action of $Gal(K|\mathbb{Q})$ on K sends elements of R to itself (hence giving an action of $Gal(K|\mathbb{Q})$ on R).

(c) (3 points) For any two integers m, n which are not divisible by p, show that the quotient $(1-\zeta^m)/(1-\zeta^n)$ is an element of R.
Hint: Reduce to the case where n divides m.

(d) (2 points) Verify that $p = (1-\zeta)\ldots(1-\zeta^{p-1})$.
Hint: manipulate the cyclotomic polynomial associated to ζ.

(e) (3 points) Prove that $1-\zeta$ is not a unit of R.

(f) (2 points) Prove (using norms) that $1-\zeta$ is an irreducible element of R. (It is true, but harder to prove, that $1-\zeta$ is in fact a prime element of R.)