Algebra general exam. August 22nd 2012, 9am-1pm

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. For a positive integer n, denote by S_n the symmetric group on $\{1, 2, \ldots, n\}$. Let $p > 2$ be a prime number.

(a) (2 pts) Give an example of a non-cyclic group of order $2p$.

(b) (5 pts) Find the smallest n for which S_n contains a cyclic subgroup of order $2p$.

(c) (7 pts) Find the smallest n for which S_n contains some subgroup of order $2p$.

In both (b) and (c), if n is your answer, explain clearly why S_n contains a desired subgroup and why S_m for $m < n$ does not contain such subgroup.

2. (8 pts) Let G be a finite group and p a prime divisor of $|G|$. Assume that every element of G of p-power order is contained in a normal p-subgroup of G. Show that G has only one Sylow p-subgroup.

3. Let k be a field and $R = k[x, y]/(x^5 - y^2)$.

(a) (5 pts) Prove that R is isomorphic to the subring $k[t^2, t^5]$ of $k[t]$ (the polynomials in one variable over k).

(b) (8 pts) Prove that R is not isomorphic to $k[t]$ (as a ring).

4. (8 pts) Let R be a commutative ring with 1. Let N be the nilradical of R, that is, N is the set of all nilpotent elements of R (including 0). You may use without proof that N is the intersection of all prime ideals of R. Prove that the following conditions are equivalent:

(i) R has just one prime ideal.

(ii) R/N is a field.

5. Recall that if R is a commutative ring with 1 and A and B are R-algebras, then $A \otimes_R B$ also has the natural structure of an R-algebra.

(a) (3 pts) Let K and L be fields of different characteristics. Prove that $K \otimes_{\mathbb{Z}} L = \{0\}$.

(b) (6 pts) Let K and L be fields of the same positive characteristic p. Prove that $K \otimes_{\mathbb{Z}} L$ can be provided in a natural way with the structure of an $\mathbb{F}_p$-algebra, and that this $\mathbb{F}_p$-algebra is isomorphic to $K \otimes_{\mathbb{F}_p} L$. Deduce that $K \otimes_{\mathbb{Z}} L$ is nonzero.

(c) (5 pts) Find an example of commutative rings A and B which are NOT fields such that $A \otimes_{\mathbb{Z}} B$ is a field. **Hint:** Use a suitable property of tensor products involving direct sums.

6. Let p be an odd prime number and n an integer ≥ 2.

(a) (5 pts) Show that $GL_n(\mathbb{Q})$ has an element of order p if and only if $n \geq p - 1$.

(b) (5 pts) Show that there is an $A \in GL_n(\mathbb{Q})$ of order p which does NOT have 1 as an eigenvalue if and only if $p - 1$ divides n.

(c) (5 pts) Let $A \in GL_4(\mathbb{Q})$ be an element of order 5. Prove that the complex JCF of A is independent of such A (up to permutation of blocks) and write it down.

7. Let K/F be a field extension, let $\alpha, \beta \in K \setminus F$ be algebraic over F, and let $p = \deg_F(\alpha)$ and $q = \deg_F(\beta)$. Suppose that p and q are distinct primes and that $p > q$.

(a) (2 pts) Prove that $[F(\alpha, \beta) : F] = pq$.

(b) (6 pts) Prove that $\deg_F(\alpha\beta) = p$ or pq.

(c) (4 pts) Give an example showing that it MAY happen that $\deg_F(\alpha\beta) = p$.

8. In this problem you may use the following fact without proof: for any group G there exists a Galois extension M/L with $Gal(M/L) \cong G$.

(a) (8 pts) Prove that there exists a field extension K/F such that $[K : F] = 4$ and there are no intermediate fields between F and K other than F and K. **Hint:** First reduce the question to a purely group-theoretic problem. Partial credit will be given for such reduction.

(b) (3 pts) Is it possible to construct an extension satisfying (a) if F is finite? Justify your answer.

(c) (5 pts) Is it possible to construct an extension satisfying (a) if K is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$ for some n (where ζ_n is a primitive n^{th} root of unity)? Justify your answer.