Algebra general exam. January 18th 2011, 9am-2pm

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let G be a group of order 30.

(a) (8 pts) Prove that G has a normal subgroup of order 3 or a normal subgroup of order 5.
(b) (6 pts) Prove that G has a normal subgroup of order 15.

2. Let p be a prime and G a finite abelian group of p-power order. Denote by

- $e(G)$ the number of elements of order p in G
- $s(G)$ the number of subgroups of order p in G
- $t(G)$ the number of subgroups of index p in G
- $h(G)$ the number of non-trivial homomorphisms from G to $\mathbb{Z}/p\mathbb{Z}$

(a) (7 pts) Prove that $s(G) = e(G)/(p-1)$ and $t(G) = h(G)/(p-1)$
(b) (7 pts) Prove that $s(G) = t(G)$. **Hint:** By (a) this is equivalent to showing that $h(G) = e(G)$. Prove the latter by expressing both quantities in terms of r, the number of summands in the elementary divisor decomposition of G.

3. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$:

(a) (5 pts) $f(x) = x^3 - 3x^2 + 15x - 7$
(b) (7 pts) $f(x) = x^5 - 3x^2 + 15x - 7$

Hint for (b): Use reduction mod 3.

4. Let R be a commutative ring with 1, and let Ω be the set of all ideals I of R such that every element of I is 0 or a zero divisor.

(a) (5 pts) Prove that Ω has a maximal element (with respect to inclusion) and moreover any element of Ω is contained in a maximal element.
(b) (8 pts) Let I be a maximal element of Ω. Prove that I must be prime.

5. For a field F and a positive integer $n \geq 2$ denote by $M_n(F)$ the set of all $n \times n$ matrices over F.

(a) (8 pts) Assume that F is algebraically closed of characteristic NOT equal to 2. Prove that for every invertible matrix $A \in M_n(F)$ there exists $B \in M_n(F)$ such that $B^2 = A$.

(b) (8 pts) Let F be a field of characteristic 2. Find an invertible matrix $A \in M_n(F)$ which cannot be written as B^2 for any $B \in M_n(F)$.

Hint (for both parts): Use the Jordan Canonical Form.

6. Let R be a commutative ring with 1. Let M, N and P be R-modules, and let $\phi : M \to N$ be an R-module homomorphism.

(a) (4 pts) Prove that there exists unique R-module homomorphism $\Phi : M \otimes_R P \to N \otimes_R P$ such that $\Phi(m \otimes p) = \phi(m) \otimes p$ for all $m \in M$ and $p \in P$.

(b) (3 pts) Assume that ϕ is surjective. Prove that Φ from part (b) is also surjective.

(c) (4 pts) Show by example that if ϕ is injective, Φ need not be injective.

7. Let p be a prime, let $\zeta_p \in \mathbb{C}$ be a primitive p^{th} root of unity and $K = \mathbb{Q}(\zeta_p)$.

(a) (7 pts) Prove that the extension $K/\mathbb{Q}$ is Galois and $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order $p-1$.

(b) (5 pts) Prove that K contains a unique subfield L of the form $\mathbb{Q}(\sqrt{m})$ where $m \in \mathbb{Z}$ and m is not a perfect square.

In parts (c)-(e) of this problem let $p = 7$, and let L denote the subfield found in part (b).

(c) (2 pts) Find (explicitly) a generator for the group $\mathrm{Gal}(K/L)$.

(d) (3 pts) Find an element $\alpha \in L \backslash \mathbb{Q}$. You may express α as a polynomial in ζ_7. **Hint:** use (c).

(e) (3 pts) Find m from part (b) explicitly (it is well defined up to multiplication by a perfect square).