Algebra general exam. January 9th, 2009.

Directions. You have **five** hours to complete this exam. Please show all your work and justify any statements that you make. You may assume the statement in an earlier part proven in order to do a leter part. DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let G be a group of order $660 = 11 \cdot 60$, and let P be a Sylow 11-subgroup of G. Assume that $C_G(P) = P$ (where C_G is the centralizer in G).

(a) (10 pts) Prove that $|N_G(P)| = 55$ (where N_G is the normalizer in G).

(b) (10 pts) Let H be a normal subgroup of G. Prove that either $P \subseteq H$ or $|H| \equiv 1 \mod 11$. **Hint:** Consider the conjugation action of P on H.

2. (a) (6 pts) Classify abelian groups of order $72 = 2^3 \cdot 3^2$ up to isomorphism (the answer is sufficient).

(b) (4 pts) Let m and n be positive integers. What is the number of elements in $\mathbb{Z}/n\mathbb{Z}$ whose order divides m?

(c) (10 pts) Let G and H be finite abelian groups, and assume that for any $m \in \mathbb{N}$ the groups G and H have the same number of elements of order m. Prove that G and H are isomorphic.

3. Let F be a field, and let R be the subring of $F[x]$ consisting of all polynomials with zero coefficient of x, that is,

$$R = \{a_0 + a_2x^2 + \ldots + a_nx^n : a_i \in F\}.$$

(a) (7 pts) Prove that the elements x^2 and x^3 are irreducible but not prime in R.

(b) (6 pts) Is R a principal ideal domain? Prove your answer.

(c) (7 pts) Prove that R is Noetherian.

4. (a) (10 pts) Prove that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain.

(b) (10 pts) Let n and m be positive integers and assume that m is a product of distinct primes. Prove that the polynomial $f(x) = x^n - m$ is irreducible in $\mathbb{Q}[x]$.

5. Let F be an algebraically closed field, and let $M_n(F)$ be the ring of $n \times n$ matrices over F.

(a) (10 pts) Prove that for any $A \in M_n(F)$ the centralizer of A has dimension at least n (**Hint:** look at each Jordan block)

(b) (10 pts) Describe all matrices $A \in M_n(F)$ with the property that every matrix commuting with A is diagonalizable.

6. The purpose of this problem is to prove that $\mathbb{R}$ has trivial group of field automorphisms.
Let ϕ be a field automorphism of $\mathbb{R}$.
(a) (6 pts) Prove that $\phi(x) = x$ for any $x \in \mathbb{Q}$.
(b) (6 pts) Prove that if $x > 0$, then $\phi(x) > 0$ as well.
(c) (8 pts) Use (a) and (b) to prove that $\phi(x) = x$ for any $x \in \mathbb{R}$.

7. Let $p(x) = x^4 - 2 \in \mathbb{Q}[x]$.
(a) (6 pts) Find a splitting field K for $p(x)$. Describe K in the form $\mathbb{Q}(\alpha, \beta)$ for some $\alpha, \beta \in \mathbb{C}$.
(b) (7 pts) Determine the Galois group $Gal(K/F)$ and describe its elements by their actions on α and β.
(c) (7 pts) Which (well-known) group is $Gal(K/F)$ isomorphic to? Prove your answer.

8. (a) (6 pts) Let R be a commutative integral domain with 1, and let I be a principal ideal of R. Prove that the R-module $I \otimes_R I$ is torsion-free, that is, if $rm = 0$ for some $r \in R$ and $m \in I \otimes_R I$, then $r = 0$ or $m = 0$.

In parts (b)-(d) of this problem let $R = \mathbb{Z}[x]$ and $I = (2, x)$, the ideal of R generated by 2 and x.

(b) (5 pts) Let $m = 2 \otimes x - x \otimes 2 \in I \otimes_R I$. Find a nonzero element $r \in R$ such that $rm = 0$.

(c) (5 pts) Consider the mappping $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ given by

$$\phi(p(x), q(x)) = \frac{p(0)}{2} q'(0) \mod 2.$$

where q' is the formal derivative of q. Prove that ϕ is R-balanced, that is, ϕ is bilinear and $\phi(rm, n) = \phi(m, rn)$ for any $r \in R$ and $m, n \in I$.

(d) (4 pts) Use (c) to prove that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$ (and thus, by (b) $I \otimes_R I$ is not torsion-free)

Reminder: You may answer (d) assuming (c) even if you failed to solve (c).