Algebra General Exam August 18, 2008

Directions: You have four hours to complete this exam. Please show all of your work, and justify any statements that you make. You may assume the statement in an earlier part proven in order to do a later part. All parts of questions are worth 4 points each. DO EACH PROBLEM ON A SEPARATE SHEET OR SHEETS OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. [12] Let A be the 5-by-5 real matrix A $:= \begin{bmatrix} 2 & 1 & 4 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

a. What is the characteristic polynomial $\chi_A(x)$ and the minimal polynomial $\mu_A(x)$?

b. Up to similarity, how many matrices $B \in \mathbb{M}_{5\times5}(\mathbb{R})$ have the same characteristic polynomial

$$\chi_B(x) = \chi_A(x)?$$

c. What is the Jordan Canonical Form of A?

2. [8] Prove the following statements about tensor products.

a. $\mathbb{Z}_5[x] \otimes_{\mathbb{Z}} \mathbb{Q}[x] = 0$.

b. $\mathbb{Q}[i] \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(i + \sqrt{2})$ as $\mathbb{Q}$-vector spaces.

3. [12] Prove or disprove the following statements about irreducibility.

a. If $Gal(E|\mathbb{Q}) = S_n$ for E the splitting field over $\mathbb{Q}$ of a polynomial $f(x) \in \mathbb{Q}[x]$ of degree n, then $f(x)$ must be irreducible.

b. If α is algebraic over $\mathbb{Q}$, then its minimum polynomial $\mu_{\alpha|\mathbb{Q}}(x)$ *must* be irreducible.

c. If $A \in \mathbb{M}_{n\times n}(\mathbb{R})$, then its minimum polynomial $\mu_A(x)$ *must* be irreducible.

4. [12] Let G be a finite abelian group,$|G| = n$,, and define the exponent of g to be $Exp(G) := \min\{k \in \mathbb{N} \mid g^k = e \textit{ for all } g \in G.\}$.

a. Prove that $Exp(G)$ divides n, and n divides $Exp(G)^j$ for some $j \in \mathbb{N}$.

b. If G, H are finite abelian groups with $Exp(G) = Exp(H)$ and $|G| = |H| = n$ where p^4 does not divide n for any prime p, prove that G and H are isomorphic.

c. State (without proof) results analogous to (a) and (b) which hold for $n\times n$ matrices over an algebraically closed field.

5. [8] Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers, and let $I = (a, b)$ be the ideal generated by $a = 16i, b = 5 + 3i$ in R.

a. Systematically find $c \in R$ such that $I = (c)$.

b. Prove or disprove: R/I is a finite field.

6. [16] Let G be a group of order pqr where p, q, r are prime numbers with $(i) p > q > r$, $(ii) \gcd(q, p-1) = \gcd(r, q-1) = \gcd(r, p-1) = 1$. [You may use without proof any general facts about Sylow subgroups proved in Math 750, 751, 752.]

a. Prove that G has a normal subgroup P of order p.

b. Prove that G has a subgroup Q of order q which commutes with P, so that $P \times Q$ is a subgroup of G of order pq.

c. Prove that Q is normal, so $P \times Q$ is a normal subgroup of G.

d. Prove that G has a subgroup R of order r which commutes with $P \times Q$, so that $P \times Q \times R$ is a subgroup of G. Conclude that G is cyclic.

7. [16] The goal of this problem is to prove that all automorphisms φ of S_5 are inner.

a. Given that the transpositions $(i, j)\,(i < j)$ generate S_n, *prove* that the adjacent transpositions $\tau_i := (i, i+1) \in S_n$ for $1 \leq i \leq n-1$ generate S_n.

b. Prove that all automorphisms of S_n leave A_n invariant, $\varphi(A_n) = A_n$.

c. Prove that the elements of order 2 in S_5 are precisely all single and double transpositions (i, j) and $(i, j)(k, \ell)$ for distinct i, j, k, ℓ, then use part (b) to show that $\varphi(\tau_i)$ is a single transposition for every automorphism $\varphi \in Aut(S_5)$.

d. Prove that every automorphism $\varphi \in Aut(S_5)$ is an inner automorphism. [Hint: count possibilities for $\varphi(\tau_i)$ to get an upper bound on the number of automorphisms of S_5.]

8. [16] This problem involves finding a seventh degree polynomial whose Galois group is isomorphic to S_7.

a. Prove that if p is prime then any transposition τ and p-cycle σ together generate all of S_p.

b. Prove that if p is a prime number, and E a splitting field over $\mathbb{Q}$ for an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ of degree p with exactly $p-2$ real roots, then the Galois group $Gal(E|\mathbb{Q}) = S_p$.

c. Give a counter-example to the statement in part (b) if the degree of the polynomial is not prime.

d. *Exhibit* (with proof) an irreducible polynomial of degree 7 over the rationals whose Galois group is S_7.