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General Exam in Algebra

Winter 2006/07

January 15, 2007

Rules

1. This is a closed book exam. To use a result, you can cite it by name (e.g., say “by the Main Theorem on Finitely Generated Modules over PIDs”) or, if the result does not have a common name, you can just restate it (e.g., say “We know from class that the polynomial ring over a UFD is a UFD”).
2. Make sure that what you write is crystal clear. It will help if you use complete sentences to communicate your ideas. Most mathematicians do not engage in reading minds. You really have to tell what is going on. Make sure that it is impossible to misunderstand your write-up.
3. If you think a problem needs clarification, please ask. The proctor will respond to the class.
4. There is a total of 50 points on this exam.
5. This exam has 10 problems and 11 pages. Make sure that none are missing in your copy.

Hints

1. Problems are not sorted according to difficulty.

Room for your pledge:

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Problem 1 [5 points]

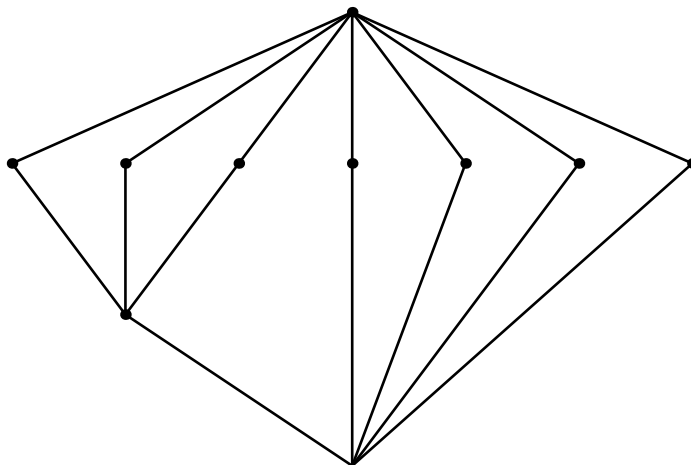
Show that every group of order 1040 admits a transitive action on a set of size 40.

Problem 2 [5 points]

Let L be a field of characteristic $\neq 2$. Show that any field extension M/L of degree 2 is Galois.

Problem 3 [5 points]

Let M/L be a Galois extension of degree 12. The following is the lattice of intermediate extensions.



Here, vertices of the same height may correspond to intermediate extensions of different degree. The top vertex corresponds to M and the bottom vertex corresponds to L .

Show that the Galois group of M/L is \mathbf{A}_4 .

Problem 4 [5 points]

Factor $x^3 - y^3$ into irreducibles over $\mathbb{Q}[x, y]$. It is important that you argue the irreducibility of the factors you find.

Problem 5 [5 points]

Show that the group $G := \mathrm{SL}_3(\mathbb{Z})$ is residually finite, i.e., the intersection of all finite index normal subgroups in G is trivial.

Extend your method and show that the group $H := \mathrm{SL}_3(\mathbb{Z}[x])$ is also residually finite.

Problem 6 [5 points]

Find all 20×20 matrices, up to similarity, over the field \mathbb{Q} with minimal polynomial $\mu(x) = (x^2 + 1)^2(x^3 + 2)$.

Problem 7 [5 points]

Let $M/L/K$ be a tower of fields, let $\alpha \in M$ be algebraic over K , and assume that M/K is a simple algebraic extension with $M = K(\alpha)$. Let $\mu_{\alpha,L}(x) = a_0 + a_1x + \cdots + a_mx^m \in L[x]$ be the minimal polynomial of α over the intermediate field L . Show that $L = K(a_0, a_1, \dots, a_m)$.

Problem 8 [5 points]

Let V be a complex vector space of finite dimension and let $\eta, \zeta : V \rightarrow V$ be two endomorphisms of V . Suppose that η and ζ commute, i.e., $\eta\zeta = \zeta\eta$. Show that η and ζ have a common eigenvector.

Problem 9 [5 points]

Prove or disprove that $L(x) \otimes_{L[x]} L[[x]] = L((x))$ for any field L .

Problem 10 [5 points]

For the following questions, no reasoning is required:

- True or false: if a group is non-abelian and simple, then all its automorphisms are inner.
- Determine $\mathbb{Z}_3 \otimes_{\mathbb{Z}} \mathbb{Z}_3$
- True or false: every noetherian UFD is a PID.
- State the universal property of a free R -module with basis X .
- True or false: a ring is an integral domain if and only if it is a subring of a field.