Algebra General Exam January 11, 2006

This is exam is worth 100 points; each problem is worth 10 points. If you can't prove part of some problem, you can still use the result of that part in proving subsequent parts or other problems. Throughout the exam, let G denote a group, V a vector space over a field F, V a vector space over F, and T a linear transformation on V.

1. A G-set $(X, \cdot_X)$ is a set with a group action $g \cdot x$ so that $g \to g\cdot$ is a homomorphism $G \to Symm(X)$ of groups. A *G-homomorphism* $X \xrightarrow{f} Y$ of G-sets satisfies $f(g \cdot_X x) = g \cdot_Y f(x)$ for all $g \in G, x \in X$. Let H be a subgroup of G; we know that the left-coset space G/H becomes a G-set under $g \cdot xH = gxH$. Describe all possible G-homomorphisms $G/H \xrightarrow{f} X$.

2. Find the number of elements of order precisely p^2 in the group $G = Z_{p^3} \times Z_{p^5}$, where Z_n denotes the cyclic group of order n.

3. If a prime p divides the order of a finite simple group G, show that $|G| < n_p!$ where n_p is the number of distinct p-Sylow subgroups of G.

4. Let R be the ring of all continuous real-valued functiuons on $[0, 1]$. Show that the set of functions f with $f(1) = 0$ is a maximal ideal of R which is not principal.

5. Let $T = \lambda Id + Z$ for λ in a field F of characteristic 0 and Z nilpotent ($Z^m = 0$ for some m). Show that if $T^k = Id$ for some $k > 0$ then Z must be the zero transformation.

6. If $T(v) \in Fv$ for all $v \in V$, show that $T = \lambda Id$ is a "scalar" for some $\lambda \in F$.

7. If G is a finite subgroup of $SL_2(\mathbb{C})$ (the complex 2×2 matrices of determinant 1) then $Av = v$ for $A \in G$, v in $\mathbb{C}^2$ implies $A = I_2$ is the 2×2 identity matrix or $v = 0$ is the zero vector.

8. For an element $a \in \mathbb{Q}$, consider the ring homomorphism φ from the polynomial ring $\mathbb{Q}[x]$ to the ring $M_3(\mathbb{Q})$ of 3×3 matrices, given by evaluation $f(x) \mapsto f(A)$ for $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$. Show that the kernel of φ is the set of all polynomials $f(x)$ with $f(a) = f'(a) = f''(a) = 0$. Is this a principal ideal?

9. Prove that $\sqrt{2} + \sqrt[3]{3}$ is irrational.

10. Find all subfields of the field $\mathbb{Q}[\zeta_3, \sqrt[3]{2}]$ for $\zeta_3 \in \mathbb{C}$ a primitive cube root of unity.