ALGEBRA GENERAL EXAM
JANUARY 10, 2004

(1) (10 point) Let $G = \{g_1, \cdots, g_n\}$ a finite abelian group. Show that the product $P := g_1 \cdots g_n$ is of order 1 or 2.

(2) (10 point) Let H be the subgroup of $G := \mathbb{Z} \oplus \mathbb{Z} = \{(m, n) | m, n \in \mathbb{Z}\}$ generated by $(1, 2)$ and $(3, 4)$. Identify the quotient group G/H.

(3) (15 point)
  (a) If $A \in M_n(\mathbb{R})$ is idempotent, i.e. $A^2 = A$, then it is diagonalizable.
  (b) Two idempotent matrices $A, B \in M_n(\mathbb{R})$ are similar if and only if they have the same rank.

(4) (10 point) Show that a principal ideal in $\mathbb{Z}[x]$ can never be maximal.

(5) (15 point) Let ξ be a primitive 9-th root of unity. Let $K = \mathbb{Q}(\xi)$ and $F = \mathbb{Q}(\xi + \xi^{-1})$.
  (a) Show that $[K : F] = 2$.
  (b) Show that the extension F/Q is normal.

(6) Let V and W be finite dimensional vector spaces over a field k, and let $f : V \to V$ and $g : W \to W$ be linear operators. One can define a linear operator $f \otimes g : V \otimes_k W \to V \otimes_k W$. Show that $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}f \cdot \mathrm{Tr}g$.

(7) (15 point) Let S be a finite set acted upon by a finite group G. Denote by $\mathbb{C}(S)$ the complex vector space of complex-valued functions on S.
  (a) Show that the map $G \times \mathbb{C}(S) \to \mathbb{C}(S)$, $(x, f) \mapsto x.f$, defines a G-action on $\mathbb{C}(S)$. Here $x.f \in \mathbb{C}(S)$ is defined by $x.f(s) = f(x^{-1}s)$ for all $s \in S$.
  (b) Show that the dimension of the subspace of G-fixed points in $\mathbb{C}(S)$ is equal to the number of G-orbits in S.

(8) (15 point) TRUE-FALSE. You do not need to justify your answers.
  (a) Every matrix in $SL_n(\mathbb{R})$ has a positive real eigenvalue if n is odd.
  (b) The Galois group of $x^3 + 1$ is isomorphic to S_3.
  (c) A group of order 99 must have a unique normal subgroup of order 11.
  (d) There exists a unique finite field of order 6 up to isomorphism.
  (e) $\mathbb{Z}[x]$ is a Euclidean domain.