ALGEBRA GENERAL EXAM
AUGUST 16, 2003, UVA

Notations. Throughout the exam, we denote by $\mathbb{Z}_n$ the cyclic group of n elements, $\mathbb{Q}$ the rational field, $\mathbb{R}$ the real field.

(1) (10 points) Let $G = SL_2(\mathbb{R})$ be the group of real 2×2 matrices with determinant one, and let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ be the upper half of the complex plane. It is known that the map $G \times \mathbb{H} \to \mathbb{H}$, $(g, z) \mapsto g(z)$, defines an action of G on $\mathbb{H}$, where

$$g(z) = \frac{az+b}{cz+d}, \qquad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G.$$

(a) Identify the stabilizer $G(i)$ of the point $z = i$ (the imaginary root) in $\mathbb{H}$ under this action.
(b) Show that the action of G on $\mathbb{H}$ is transitive.

(2) (10 points) Let D be a PID with F as its field of fractions. Show that every element $x \in F$ can be written as a sum of *primary fractions* (i.e. with denominators powers of primes):

$$x = \sum_{i=1}^{n} \frac{a_i}{p_i^{e_i}}$$

for some $a_1, \ldots, a_n \in D$, and distinct primes $p_1, \ldots, p_n \in D$.

(3) (15 points) Let P_{2n-1} be the vector space of polynomials in one variable x with real coefficients of degree $\leq 2n-1$. Let T denote the linear operator on P_{2n-1} defined by $p(x) \mapsto p(x) + p''(x)$ for every $p(x) \in P_{2n-1}$, where p'' denotes the second derivative.
(a) Write the matrix A_T of the operator T with respect to the basis $\{1, x, x^2, \ldots, x^{2n-1}\}$ of P_{2n-1}.
(b) Find the Jordan canonical form of A_T AND a Jordan basis for T.

(4) (15 points)
(a) Determine the following tensor products and explain your answers.
(i) $\mathbb{R} \bigotimes_{\mathbb{Z}} \mathbb{Z}_{2003}$
(ii) $\mathbb{Q}[x]/(x^2+1) \bigotimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x^2+2)$.
(b) Let V and W be finite-dimensional vector spaces over a field F, and $\{v_1, \ldots, v_n\}$ be a basis of V. Prove that if

$$v_1 \otimes w_1 + \cdots + v_n \otimes w_n = 0$$

in $V \bigotimes_F W$ for $w_1, \ldots, w_n \in W$, then $w_1 = \cdots = w_n = 0$.

(5) (10 points) Choose your favorite one, denoted by G, between the Klein four group (i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$) and the diheral group D_4 of order 8. Provide an example of an irreducible degree 4 polynomial whose Galois group over $\mathbb{Q}$ is isomorphic to G. Show your work.

(6) (15 points) Let B be a symmetric bilinear form on a finite-dimensional vector space V over a field F. For a subspace $W \subset V$, we define the annihilator subspace of W in V: $W^\perp = \{x \in V \mid B(x, w) = 0 \text{ for every } w \in W\}$. Assume further that B is nondegenerate, that is $V^\perp = \{0\}$. Show that:
(a) $\dim W^\perp = \dim V - \dim W$.
(b) $(W^\perp)^\perp = W$.

(7) (10 points for (a) and (b)) (Hint: proof by contradiction could be useful)
Let G be a finite group of order n. Suppose that for every d dividing n, the equation $x^d = 1$ has at most d solutions in G. Show that:
(a) For each prime p, the Sylow p-subgroup of G is unique, and thus is normal.
(b) The Sylow p-subgroup of G is cyclic.
(c) (**bonus problem**) Use (a) and (b) to show that G is cyclic.

(8) (15 points) Determine whether each of the following statements is TRUE or FALSE. You do NOT need to show your work.
(a) Let G be a finite group. Assume that H is a normal subgroup of G and K is a normal subgroup of H. Then K is normal in G.
(b) If R is PID, so is $R[x]$.
(c) The *integer orthogonal group* $O_n(\mathbb{Z})$, which consists of $n \times n$ orthogonal matrices whose entries are all integers, is a finite group.
(d) Let p be a prime. Then the number of monic irreducible polynomials of degree 2 over a finite field $\mathbb{F}_p$ is $p(p-1)/2$.
(e) If an arbitrary integer polynomial $f(x)$ has no roots in $\mathbb{Z}$, then it has no roots in $\mathbb{Q}$.