General Exam in Algebra

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1. Let $A \in M_n(\mathbb{R})$ be an alternating matrix, i.e. ${}^tA = -A$. Show that if n is odd then $\det A = 0$.
2. Let $A, B \in M_n(\mathbb{C})$ be two commuting matrices. Prove that they have a common eigenvector in $\mathbb{C}^n$, i.e. there exists a nonzero $v \in \mathbb{C}^n$ such that $Av = \lambda v$ and $Bv = \mu v$ for some $\lambda, \mu \in \mathbb{C}$.
3. Let $U = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \ : \ a_{ij} \in \mathbb{R} \right\}$. Prove that U is a group for multiplication of matrices and identify the center $Z(G)$ and the commutator subgroup $[G, G]$ (we recall that $[G, G]$ is the subgroup generated by all commutators $xyx^{-1}y^{-1}$ with $x, y \in G$.
4. Show that the group G_1 of all real numbers for addition is isomorphic to the group G_2 of all positive real numbers for multiplication. Furthermore, show that the group H_1 of all rational numbers is **not** isomorphic to the group H_2 of all positive rational numbers for multiplication.
5. Let G be a group such that there exists a surjective group homomorphism $G \to \mathbb{Z}$. Prove that for any subgroup of finite index $H \subset G$ there also exists a surjective group homomorphism $H \to \mathbb{Z}$.
6. Let A be the ring of all continuous real-valued functions on $[0, 1]$. Give an example of a maximal ideal in A. Furthermore, give an example of an element $f \in A$ which is not invertible, but which is not a zero divisor either.
7. Let $f(x) \in \mathbb{R}[x]$ be a nonzero polynomial. Show that there exists a number $r \in \mathbb{R}$ such that the polynomials $f(x)$ and $f(x + r)$ are relatively prime.
8. Let $K = \mathbb{Z}/p\mathbb{Z}$ be the field of p elements (where p is a prime). For an integer $d > 0$, we let

$$\sigma_d = \sum_{x \in K} x^d.$$

Show that

$$\sigma_d = \begin{cases} -1 & \text{if } (p-1) \text{ divides } d \\ 0 & \text{otherwise} \end{cases}$$