

Topology General Exam Syllabus

Revised summer 2012

I. Differential Topology

- (1) Multivariable calculus basics: definition of a smooth map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the inverse and implicit function theorems.
- (2) Manifolds and smooth maps; submanifolds. Examples: 2-dimensional surfaces; the sphere S^n ; the real projective space $\mathbb{R}P^n$; examples of Lie groups: classical matrix groups.
- (3) The differential of a smooth map, tangent vectors, and tangent spaces. The tangent bundle.
- (4) Regular and critical values. Embeddings, immersions. Transversality.
- (5) Sard's theorem.
- (6) The embedding theorem: every closed manifold embeds in a Euclidean space.
- (7) Orientability.
- (8) Vector fields, the Euler characteristic.
- (9) Invariants of manifolds and smooth maps: mod 2 degree of a map, the integer-valued degree of a map between oriented manifolds, intersection numbers, linking numbers.
- (10) Applications to compact manifolds: ∂M is not a retract of M , Brouwer Fixed Point Theorem, zeros of vector fields, etc.

II. Algebraic Topology

- (1) Basic properties of singular homology: functoriality, homotopy invariance, long exact sequence of a pair, excision, and the Mayer–Vietoris sequence.
- (2) Homological algebra: chain complexes, maps, homotopies. The long exact homology sequence associated to a s.e.s. of chain complexes. The snake lemma. The 5-lemma.
- (3) The homology groups of spheres, and the degree of a map between spheres. Classic applications, as in I (10) above, but proved with homology.
- (4) The Jordan–Alexander Complement Theorem: $H_*(\mathbb{R}^n - A) \simeq H_*(\mathbb{R}^n - B)$ if A and B are homeomorphic closed subsets of \mathbb{R}^n . The Jordan Curve Theorem is a special case.
- (5) The homology of a C.W. complex: cellular chains. This includes delta complex homology as a special case. Examples: real and complex projective spaces, closed surfaces.
- (6) Euler characteristic and its properties. Classic calculations: spheres, closed surfaces.
- (7) Construction of the fundamental group as a homotopy functor of a space with basepoint.
- (8) Covering spaces: definition and examples.
- (9) The lifting theorem: under appropriate point set conditions, a continuous $f : X \rightarrow Y$ lifts through a covering map $\tilde{Y} \rightarrow Y$ iff it does on the level of π_1 .
- (10) Deck transformations, and the correspondence between subgroups of the fundamental groups and covering spaces. A variant: if \tilde{Y} is simply connected, there is a 1–1 correspondence between covering spaces $\tilde{Y} \rightarrow Y$ and free, proper group actions on \tilde{Y} .
- (11) Seifert–Van Kampen Theorem.
- (12) H_1 is the abelianization of π_1 .
- (13) Examples, including the fundamental group of spheres, projective space, surfaces, etc. Classic applications, e.g., to group theory.

References:

- **Differential Topology** by V.Guillemin and A.Pollack.
- **Differential topology: first steps** by A.Wallace
- **Topology from a Differential Viewpoint** by J.Milnor
- **Algebraic Topology** by A.Hatcher.
- **Homology Theory** by J.W.Vick (2nd edition).

References for general topology background material:

- **Topology** by J.R.Munkres.
- *An outline summary of basic point set topology*, by J.P.May,
at <http://www.math.uchicago.edu/~may/MISC/Topology.pdf>

References for the Jordan–Alexander Complement Theorem:

- A Dold, *A simple proof of the Jordan–Alexander Complement Theorem*, Amer. Math. Monthly **100** (1993), 856–857.
- Both Hatcher and Vick prove special cases including the Jordan Curve Theorem.