Topology General Exam Syllabus

Revised February 2022

I. Differential Topology

(1) Multivariable calculus basics: definition of a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, the inverse and implicit function theorems.
(2) Manifolds and smooth maps; submanifolds. Examples: 2-dimensional surfaces; the sphere S^n; the real projective space $\mathbb{R}P^n$; examples of Lie groups: classical matrix groups.
(3) The differential of a smooth map, tangent vectors, and tangent spaces. The tangent bundle.
(4) Regular and critical values. Embeddings, immersions. Transversality.
(5) Sard's theorem.
(6) The embedding theorem: every closed manifold embeds in a Euclidean space.
(7) Orientability.
(8) Vector fields, the Euler characteristic.
(9) Invariants of manifolds and smooth maps: mod 2 degree of a map, the integer-valued degree of a map between oriented manifolds, intersection numbers, linking numbers.
(10) Applications to compact manifolds: ∂M is not a retract of M, Brouwer Fixed Point Theorem, zeros of vector fields, etc.
(11) Vector bundles: tangent bundle, normal bundle, duals, tensor bundles. Structures on bundles including inner products, specifically Riemannian metrics.
(12) Differential forms: exterior algebra, exterior derivative.
(13) Integration of forms on oriented manifolds; Riemannian volume form; Stokes' theorem.

II. Algebraic Topology

(1) Basic properties of singular homology: functoriality, homotopy invariance, long exact sequence of a pair, excision, and the Meyer–Vietoris sequence.
(2) Homological algebra: chain complexes, maps, homotopies. The long exact homology sequence associated to a s.e.s. of chain complexes. The snake lemma. The 5–lemma.
(3) The homology groups of spheres, and the degree of a map between spheres. Classic applications, such as Brouwer fixed point theorem, but proved with homology.
(4) The Jordan–Alexander Complement Theorem: $H_*(\mathbb{R}^n - A) \cong H_*(\mathbb{R}^n - B)$ if A and B are homeomorphic closed subsets of $\mathbb{R}^n$. The Jordan Curve Theorem is a special case.
(5) The homology of a C.W. complex: cellular chains. This includes delta complex homology as a special case. Examples: real and complex projective spaces, closed surfaces.
(6) Euler characteristic and its properties. Classic calculations: spheres, closed surfaces.
(7) Construction of the fundamental group as a homotopy functor of a space with basepoint.
(8) Covering spaces: definition and examples.
(9) The lifting theorem: under appropriate point set conditions, a continuous $f : X \to Y$ lifts through a covering map $\tilde{Y} \to Y$ iff it does on the level of π_1.
(10) Deck transformations, and the correspondence between subgroups of the fundamental groups and covering spaces. A variant: if $\tilde{Y}$ is simply connected, there is a 1–1 correspondence between covering spaces $\tilde{Y} \to Y$ and free, proper group actions on $\tilde{Y}$.
(11) Seifert–Van Kampen Theorem.
(12) H_1 is the abelianization of π_1.
(13) Examples, including the fundamental group of spheres, projective space, surfaces, etc. Classic applications, e.g., to group theory.

References:

- *An Introduction to Manifolds* by L. Tu.
- *Differential Topology* by V. Guillemin and A. Pollack.
- *Introduction to Smooth Manifolds* by J. Lee
- *Algebraic Topology* by A. Hatcher.
- *Geometry and Topology* by G. Bredon
- *Homology Theory* by J. W. Vick (2nd edition).

References for general topology background material:

- *Topology* by J. R. Munkres.
- An outline summary of basic point set topology, by J.P.May, at `http://www.math.uchicago.edu/~may/MISC/Topology.pdf`

References for the Jordan-Alexander Complement Theorem:

- A Dold, A simple proof of the Jordan-Alexander Complement Theorem, *Amer. Math. Monthly* **100** (1993), 856–857.
- Both Hatcher and Vick prove special cases including the Jordan Curve Theorem.