

# Syllabus for Algebra General Exam

This (“new”) syllabus is a revised and updated version of the existing (“old”) syllabus. Some new topics (such as symmetric polynomials, basics of representation theory of finite groups, elements of category theory) have been added, and more importantly, the focus has been somewhat shifted from certain topics to others in order to adequately reflect current trends in algebra. While these changes will help to better equip students with fundamental algebraic tools needed to navigate modern mathematics and streamline the coverage of general exams in algebra, the new syllabus will not take effect until May 2024 after the material has been taught in the graduate algebra sequence in AY 2023-24. So, the generals in algebra in August 2023 and January 2024 will be given based on the old syllabus.

The material in the *Standard* category will be taught in Algebra I-II every year; it constitutes the basis for general exams in algebra. The material in the *Additional topics* category will be taught occasionally; not more than two of the additional topics taught in a particular year may be included in the syllabus of the general exams in the following year - these topics will be announced in May upon the completion of the academic year. This syllabus does not list the material which is considered to be part of the standard undergraduate curriculum (like groups and their (normal) subgroups, group homomorphisms and automorphisms, quotients and isomorphism theorems etc).

## I Groups

*Standard material.* Group actions, orbits and stabilizers. Regular action and action on cosets; Cayley’s theorem. Conjugation action: conjugacy classes, centralizers and normalizers. Class equation. Finite  $p$ -groups and their properties. Sylow theorems: proof and applications. Direct and semi-direct products. Groups of permutations: conjugacy in  $S_n/A_n$  and the simplicity of  $A_n$  for  $n \geq 5$ . Solvable and nilpotent groups. Free groups.

*Additional topics.* Defining groups by generators and relators (dihedral groups, Coxeter groups). Classical groups.

## II Rings

*Standard material.* Zero-divisors and domains. Left, right and two-sided ideals. Prime and maximal ideals of commutative rings. Localization with respect to a multiplicative set; local rings. Nilradical. Unique factorization domains. Unique factorization in principal ideal domains. Euclidean domains. The ring of Gaussian integers and its arithmetic. Gauss’ lemma and unique factorization in polynomial rings of several variables. Symmetric polynomials; Vieta’s formulas and the discriminant. Noetherian rings and the Hilbert’s Basis Theorem.

*Additional topics.* Basics of noncommutative rings and algebras (definitions and examples: group algebras, quaternions). The Artin-Wedderburn theory of semi-simple rings.

## III Modules and Advanced Linear Algebra

*Standard material.* Modules, submodules, homomorphisms, quotients and isomorphism theorems. Exact sequences and commutative diagrams. Free modules and projective modules. Noetherian modules. Structure of finitely generated modules over a principal ideal domain; applications: finitely generated abelian groups and canonical forms of matrices. Tensor products of modules over commutative rings. Algebras over a field and their tensor products. Tensor algebra.

The Jordan canonical and the rational canonical forms; applications. Minimal polynomial. Criteria for diagonalizability.

Introduction to the representation theory of finite groups: complete reducibility of complex representations; characters and character tables; examples.

*Additional topics.* Injective and flat modules. Basics of category theory. Linear algebra in inner product spaces. Exterior and symmetric algebras.

#### **IV Fields and Galois theory**

*Standard material.* Finite and algebraic extensions of fields. Algebraic closure. The splitting field of a polynomial; normal extensions. Separable extensions. Primitive element theorem. Finite fields. Norms and traces in finite extensions. Fundamental theorem of Galois theory and its applications. Cyclotomic extensions. Computation of Galois groups.

*Additional topics.* Kummer theory. Artin-Schreier extensions. Solvability by radicals. Inseparable extensions.