Heisenberg categories, towers of algebras, and up/down transition functions

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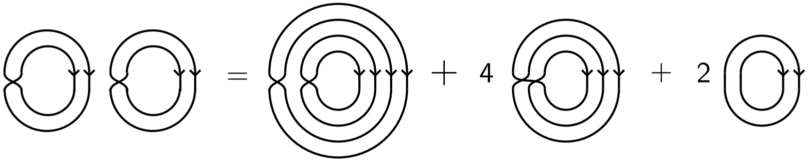
Workshop in rep. theory, combinatorics, and geometry
UVA 2018



Coherent systems on graded graphs

Borodin-Olshanski identified formalism for situation where there is a probability measure M_n on each subset G_n of a graded set $G = \bigcup_{n \geq 0} G_n \ldots$

$\vdots \qquad \vdots \qquad \vdots$

$M_4 \qquad L^{4,1} \quad L^{4,2} \quad L^{4,3} \quad L^{4,4} \quad L^{4,5}$

$M_3 \qquad L^{3,1} \quad L^{3,2} \quad L^{3,3}$

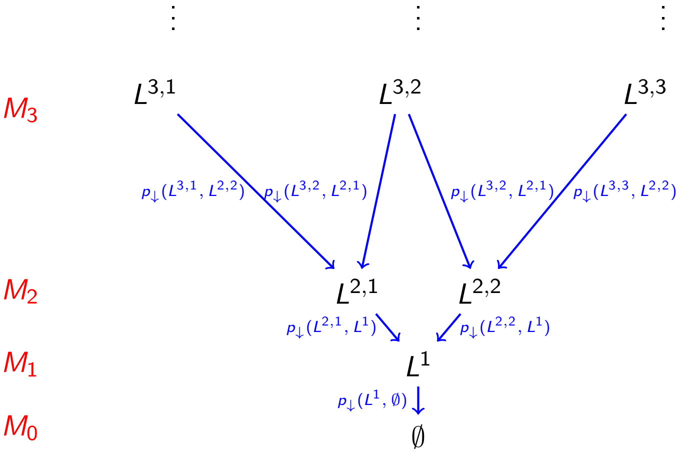
$M_2 \qquad L^{2,1} \quad L^{2,2}$

$M_1 \qquad L^{1}$

$M_0 \qquad \emptyset$

Coherent systems on graded graphs

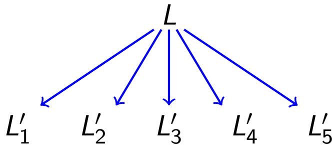
... and some *down-transition function* $p_\downarrow$ giving a Markov transition kernel between different levels of this graph



Coherent systems on graded graphs

The down transition function $p_{\downarrow} : G \times G \to [0,1]$ satisfies some properties:

1. $p_{\downarrow}(L, L') = 0$ unless $L \in G_n$, $L' \in G_{n-1}$,
2. For fixed $L \in G_n$, $\sum_{L' \in G_{n-1}} p_{\downarrow}(L, L') = 1$.



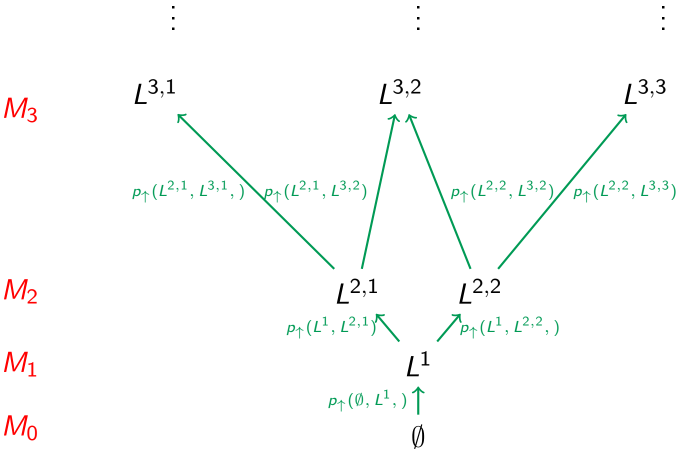
$$p_{\downarrow}(L, L'_1) + p_{\downarrow}(L, L'_2) + p_{\downarrow}(L, L'_3) + p_{\downarrow}(L, L'_4) + p_{\downarrow}(L, L'_5) = 1$$

The collection $\{M_n\}_{n \geq 0}$ is said to be *coherent* with respect to $p_{\downarrow}$ if

$$\sum_{L \in G_n} M_n(L) p^{\downarrow}(L, L') = M_{n-1}(L').$$

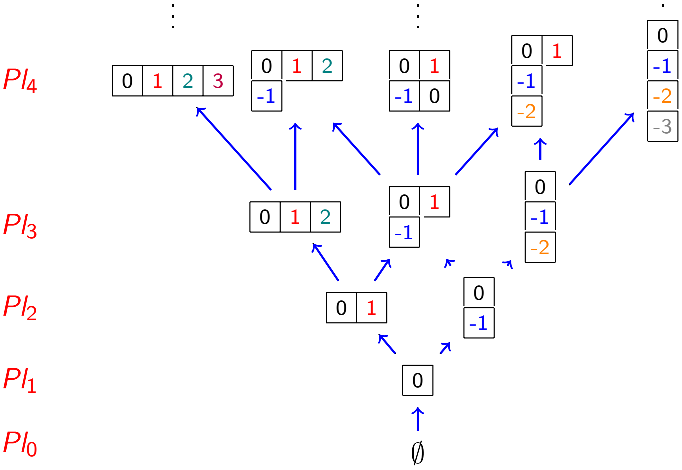
Coherent systems on graded graphs

In this case we can define an *up-transition function* $p_\uparrow : G \times G \to [0,1]$, with similar coherent properties.



A coherent system on $\mathbb{Y}$

Primary example: Take $L = \mathbb{Y}$, the set of Young diagrams, and $M_n = Pl_n$ the Plancherel measure on each level.



A coherent system on $\mathbb{Y}$

Primary example: Recall Plancherel measure for symmetric group S_n

$$Pl_n(\mu) := \frac{\dim(L^\mu)^2}{\dim(\mathbb{C}[S_n])} \qquad \mu \in \mathbb{Y}_n, \quad \Big(\text{Recall } \mathbb{C}[S_n] \cong \bigoplus_{\mu\in\mathbb{Y}_n} (L^\mu)^{\oplus \dim(L^\mu)}\Big)$$

$\{Pl_n\}_{n\geq 0}$ is coherent with respect to down-transition function

$$p_\downarrow(\mu,\eta) = \frac{\dim(L^\eta)}{\dim(\mathrm{Res}^n_{n-1} L^\mu)} = \frac{\dim(L^\eta)}{\dim(L^\mu)} \qquad \text{if } L^\eta \subseteq \mathrm{Res}^n_{n-1} L^\mu,$$

and up-transition function

$$p_\uparrow(\mu,\lambda) = \frac{\dim(L^\lambda)}{\dim(\mathrm{Ind}^{n+1}_n L^\mu)} = \frac{\dim(L^\lambda)}{n\dim(L^\mu)} \qquad \text{if } L^\lambda \subseteq \mathrm{Ind}^{n+1}_n L^\mu,$$

A coherent system on $\mathbb{Y}$

By an observation of Biane, data associated to $p_\downarrow$ and $p_\uparrow$ is captured by a family of elements in the center of $Z(\mathbb{C}[S_n])$ constructed from Jucys-Murphy elements $\{J_n\}_{n\geq 0}$.

Set $E_{n+1,n} : \mathbb{C}[S_{n+1}] \to \mathbb{C}[S_n]$ to be the map so that for $g \in S_{n+1}$,

$$E_{n+1,n}(g) := \begin{cases} g & \text{if } g \in S_n \\ 0 & \text{otherwise} \end{cases}$$

and

1. $\displaystyle\sum_{g \in S_n/S_{n-1}} g J_n^k g^{-1} \in Z(\mathbb{C}[S_n])$,
2. $E_{n+1,n}(J_{n+1}^k) \in Z(\mathbb{C}[S_n])$.

A coherent system on $\mathbb{Y}$

For $\mu \in \mathbb{Y}_n$, $\widetilde{\chi}^\mu := \frac{\chi^\mu}{\dim(L^\mu)}$ the normalized character for L^μ,

$$\widetilde{\chi}^\mu\Big(\sum_{g \in S_n/S_{n-1}} g J_n^k g^{-1} \Big) = n \sum_{\eta \in \mathbb{Y}_{n-1}} p_\downarrow(\mu, \eta)(\alpha_\eta^\mu)^k =: n m_k^\downarrow(\mu)$$

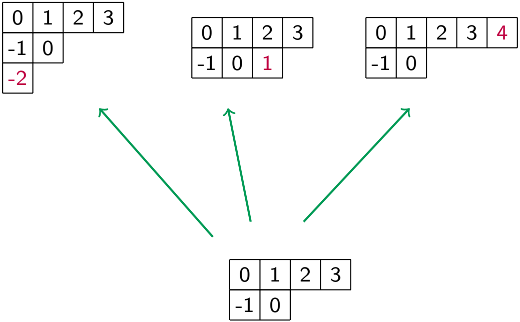
and

$$\widetilde{\chi}^\mu(E_{n+1,n}(J_{n+1}^k)) = \sum_{\eta \in \mathbb{Y}_{n+1}} p_\uparrow(\mu, \lambda)(\alpha_\mu^\lambda)^k =: m_k^\uparrow(\mu).$$

Here α_μ^λ is the eigenvalue for J_{n+1} on $L^\mu \subseteq \mathrm{Res}_{n-1}^n L^\lambda$ (combinatorially, just content of cell we remove from λ to get μ).

A coherent system on $\mathbb{Y}$

Example:



$$m_k^{\uparrow}(4,2) = \frac{\dim(L^{(4,2,1)})}{7\dim(L^{(4,2)})}(-2)^k + \frac{\dim(L^{(4,3)})}{7\dim(L^{(4,2)})}(1)^k + \frac{\dim(L^{(5,2)})}{7\dim(L^{(4,2)})}(4)^k$$

$$= p_{\uparrow}(L^{(4,2)}, L^{(4,2,1)})(-2)^k + p_{\uparrow}(L^{(4,2)}, L^{(4,3)})(1)^k + p_{\uparrow}(L^{(4,2)}, L^{(5,2)})(4)^k$$

A coherent system on strict partitions

Due to work of Borodin, Petrov, and collaborators a similar story holds when:

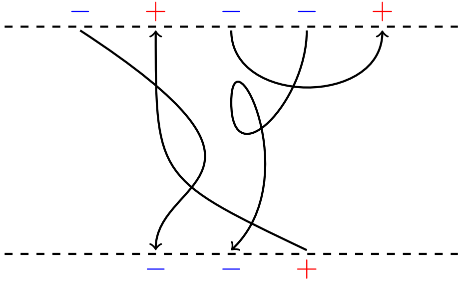
- $\mathbb{C}[S_n] \to$ Sergeev algebra $\mathbb{S}_n$,
- $L =$ set of strict partitions,
- $M_n =$ Plancherel measure on $\mathbb{S}_n$,
- For $y \in \mathbb{S}_{n+1}$, $E_{n+1,n}(y) = \begin{cases} y & \text{if } x \in \mathbb{S}_n \\ 0 & \text{otherwise,} \end{cases}$
- $J_n =$ the analogue of Jucys-Murphy elements in $\mathbb{S}_n$.

This phenomenon exists in some generality?

Heisenberg categories

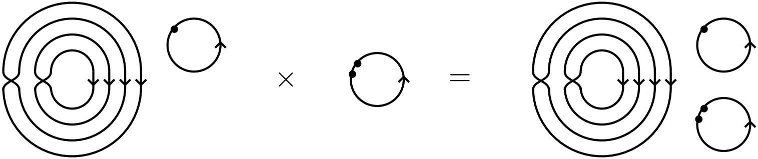
Heisenberg categories are diagrammatically defined monoidal categories which (conjecturally) categorify infinite dimensional Heisenberg algebras.

- Usually a categorical action of $\mathcal{H}$ on some $\bigoplus_{n \geq 0} A_n$-mod.
- This action gives a surjective map from $Z(\mathcal{H})$ to $Z(A_n)$ for any $n \geq 0$.



Heisenberg categories

By definition the center $Z(\mathcal{H})$ of $\mathcal{H}$ is graphically the commutative $\mathbb{C}$-algebra of all closed diagrams.



$\mathcal{H}$ is **rich** in representation-theoretic data (morphism spaces contain all symmetric groups, affine degenerate Hecke algebras).

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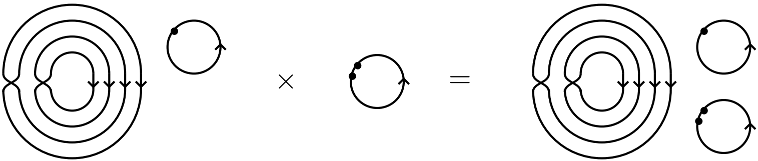


$\mathcal{H}$ is **rich** in representation-theoretic data (morphism spaces contain all symmetric groups, affine degenerate Hecke algebras).

$$\Downarrow$$

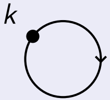
$Z(\mathcal{H})$ should contain interesting information.

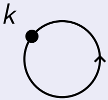
Center of $\mathcal{H}$ associated with $\{\mathbb{C}[S_n]\}_{n\geq 0}$

Theorem (Khovanov)

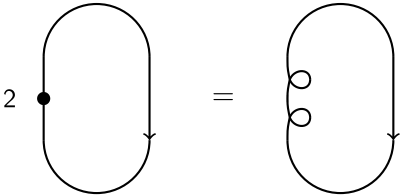
$$Z(\mathcal{H}) \cong \mathbb{C}[c_0, c_1, \ldots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \ldots]$$

where

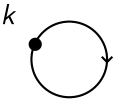
$c_k \longmapsto$ 

$\tilde{c}_k \longmapsto$ 

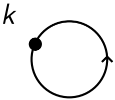
In our notation for $\mathcal{H}$ a dot labelled with a k is k right-twisted curls.



Then



$\longmapsto \quad p_1 \, m_k^{\downarrow}$



$\longmapsto \quad m_k^{\uparrow}$

Describing $\mathcal{H}$

How general is this phenomenon?

Heisenberg category	Tower of algebras	bubbles $\overset{?}{\longleftrightarrow}$ $m_k^{\downarrow}, m_k^{\uparrow}$
Khovanov's Heisenberg category	symmetric groups $\{S_n\}$	Yes, (K.-Licata-Mitchell)
spin Heisenberg category (Cautis-Sussan)	Sergeev algebras $\{\mathbb{S}_n\}$	Yes (K.-Oğuz-Reeks)
$\mathcal{H}_F$ (Rosso-Savage)	Frobenius wreath product algebras $\{F^{\otimes n} \rtimes S_n\}$	*in progress*
higher Heisenberg categories (Mackaay-Savage)	degenerate cyclotomic Hecke algebras $\{H_n^{\lambda}\}$	*in progress*

A generalization to free Frobenius towers

First question: How broad is this connection between "bubbles" in a Heisenberg category and up/down-transition functions on representations of the associated tower of algebras $\{A_n\}_{n\geq 0}$.

In particular, there are Heisenberg categories associated to towers $\{A_n\}_{n\geq 0}$ where A_n is **not semisimple**.

Next question: What is the analogue of this "Plancherel" type coherent system when algebras $\{A_n\}_{n\geq 0}$ are not semisimple?

Strategy: Follow the algebra…

A generalization to free Frobenius towers

While for a semisimple $\mathbb{C}$-algebra A_n with simple representations $\{L^\lambda\}_{\lambda \in \Gamma_n}$, as a left A_n-mod

$$A_n \cong \bigoplus_{\lambda \in \Gamma_n} (L^\lambda)^{\oplus \dim(L^\lambda)} \quad \Longleftrightarrow \quad Pl_n(\lambda) = \frac{\dim(L^\lambda)^2}{\dim(A_n)}.$$

When A_n is not semisimple we instead have

$$A_n \cong \bigoplus_{\lambda \in \Gamma_n} (P^\lambda)^{\oplus \dim(L^\lambda)} \quad \Longleftrightarrow \quad \widetilde{Pl}_n(\lambda) = \frac{\dim(L^\lambda) \dim(P^\lambda)}{\dim(A_n)}.$$

For P^λ the projective cover of L^λ.

In the non-semisimple case we see a hidden duality between simple modules and indecomposable projective modules.

A generalization to free Frobenius towers

What are the natural up/down-transition functions associated to this "new" Plancherel measure?

We should pass to Grothendieck groups and see what induction and restriction tells us in world of simples and projectives,

$$K_0(A) = \bigoplus_{n \geq 0} K_0(A_n\text{-}\mathrm{Pmod}).$$

and

$$G_0(A) = \bigoplus_{n \geq 0} G_0(A_n\text{-}\mathrm{mod}).$$

A generalization to free Frobenius towers

Functors Ind_n^{n+1} and Res_n^{n+1} descend to linear operators:

$$\mathrm{Ind}_n^{n+1}[P^\mu] = \sum_{\lambda \in \Gamma_{n+1}} \kappa(\mu, \lambda)[P^\lambda],$$

$$\mathrm{Ind}_n^{n+1}[L^\mu] = \sum_{\lambda \in \Gamma_{n+1}} \kappa^*(\mu, \lambda)[L^\lambda],$$

$$\mathrm{Res}_n^{n+1}[P^\lambda] = \sum_{\mu \in \Gamma_n} \bar{\kappa}(\lambda, \mu)[P^\mu],$$

$$\mathrm{Res}_n^{n+1}[L^\lambda] = \sum_{\mu \in \Gamma_n} \bar{\kappa}^*(\lambda, \mu)[L^\mu].$$

A generalization to free Frobenius towers

If $\kappa(\mu,\lambda)$, $\kappa^*(\mu,\lambda)$, $\bar{\kappa}(\lambda,\mu)$, $\bar{\kappa}^*(\lambda,\mu)$ are all different this is a mess...

But if Ind_n^{n+1} and Res_n^{n+1} are biadjoint

$$\Updownarrow$$

$\{A_n\}_{n\geq 0}$ is a tower of Frobenius extensions (a *Frobenius tower*) and A_{n+1} is a free (A_n, A_n)-bimodule (a *Frobenius tower*),

then things are simpler... $\Big(\kappa(\mu,\lambda) = \bar{\kappa}^*(\lambda,\mu)$ and $\kappa^*(\mu,\lambda) = \bar{\kappa}(\lambda,\mu)\Big)$.

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Take away: free Frobenius towers are a good place to look for generalizations.

A generalization to free Frobenius towers

What are the transition functions for free Frobenius tower $\{A_n\}_{n \geq 0}$? There are at least two choices:

$$p_{\downarrow}(\lambda, \mu) = \frac{\kappa(\mu, \lambda)\, \mathrm{dim}(L^{\mu})}{\mathrm{dim}(L^{\lambda})} \quad \text{and} \quad p_{\uparrow}(\mu, \lambda) = \frac{\mathrm{dim}(A_n)}{\mathrm{dim}(A_{n+1})} \frac{\kappa(\mu, \lambda)\, \mathrm{dim}(P^{\lambda})}{\mathrm{dim}(P^{\mu})}.$$

and

$$p^*_{\downarrow}(\lambda, \mu) = \frac{\kappa^*(\mu, \lambda)\, \mathrm{dim}(P^{\mu})}{\mathrm{dim}(P^{\lambda})} \quad \text{and} \quad p^*_{\uparrow}(\mu, \lambda) = \frac{\mathrm{dim}(A_n)}{\mathrm{dim}(A_{n+1})} \frac{\kappa^*(\mu, \lambda)\, \mathrm{dim}(L^{\lambda})}{\mathrm{dim}(L^{\mu})}.$$

Do the centers $\{Z(A_n)\}_{n \geq 0}$ prefer one choice? Possibly...

A generalization to free Frobenius towers

The Frobenius tower structure on $\{A_n\}_{n\geq 0}$ gives *Frobenius homomorphism*, (A_k, A_k)-bimodule homomorphism for $k < n$:

$$E_{n,k} : A_n \to A_k$$

and *dual bases* $B_{n,k}$, $B^{\vee}_{n,k}$,

$$\left(a = \sum_{b \in B_{n,k}} E_{n,k}(ab^{\vee})b = \sum_{b \in B_{n,k}} b^{\vee} E_{n,k}(ba).\right)$$

A generalization to free Frobenius towers

In the case of $\{\mathbb{C}[S_n]\}_{n\geq 0}$ and $\{\mathbb{S}_n\}_{n\geq 0}$. We captured data for $p_{\downarrow}$ and $p_{\uparrow}$ using central elements involving some version of Jucys-Murphy elements.

Assume then that $\{A_n\}_{n\geq 0}$ has a sequence of "Jucys-Murphy"-type elements $\{x_n\}_{n\geq 0}$, with

- $x_n \in A_n$,
- x_n commutes with A_{n-1},
- constant generalized eigenvalue α_η^μ on $P^\eta \subseteq \mathrm{Res}_{n-1}^n P^\mu$,
- and several other technical conditions...

A generalization to free Frobenius towers

The miracle is that up/down-transition data for $p^*_\downarrow$ and $p^*_\uparrow$ is encoded by elements in $\{Z(A_n)\}_{n\geq 0}$. Set

$$\sum_{b\in B_{n+1,0}} b^\vee x_n^k b \in Z(A_n)$$

and

$$E_{n+1,n}\Big(\sum_{b\in B_{n+1,0}} b^\vee b\, x_{n+1}^k\Big) \in Z(A_n).$$

A generalization to free Frobenius towers

Generalizing, for $\mu \in \Gamma_n$,

$$\widetilde{\chi}^{\mu}(\cdot) : A_n \to \mathbb{C} \quad \Longrightarrow \quad E_{n,0}\Big(\frac{e_\mu}{\dim(P^\mu)} \cdot \Big) : A_n \to \mathbb{C}$$

we have

$$E_{n,0}\Big(\frac{e_\mu}{\dim(P^\mu)} \sum_{b \in B_{n+1,0}} b^\vee x_n^k b\Big) = \frac{\dim(A_{n+1})}{\dim(A_n)} \sum_{\eta \in \Gamma_{n-1}} p_\downarrow^*(\mu, \eta)(\alpha_\eta^\mu)^k.$$

and

$$E_{n,0}\Big(\frac{e_\mu}{\dim(P^\mu)} E_{n+1,n}(\sum_{b \in B_{n+1,0}} b^\vee b\, x_{n+1}^k)\Big) = \sum_{\lambda \in \Gamma_{n+1}} p_\uparrow^*(\mu, \lambda)(\alpha_\mu^\lambda)^k.$$

Future directions

Towers that would interesting to study:

1. wreath product algebras $\{(F^{\otimes n} \rtimes S_n)\}_{n\geq 0}$ with F a Frobenius graded superalgebra (interesting examples: zig-zag algebra).
2. cyclotomic affine degenerate Hecke algebras $\{H_n^{\lambda}\}_{n\geq 0}$.

Hope: We may be able to prove things in these more exotic towers where combinatorics is difficult by working with central elements/Heisenberg category diagrammatics?

Thank you!