Kirillov-Reshetikhin Crystals and Cacti

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October 20, 2018

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of trace 0 matrices and $V = \mathbb{C}^n$. The standard basis vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ form a basis for V that has several good properties:

1. Each basis vector is an eigenvector for the action of the subalgebra $\mathfrak{h}$ of diagonal matrices, i.e. $\mathrm{diag}(t_1, \ldots t_n) \cdot \mathbf{v_k} = t_k \mathbf{v_k}$
2. The matrices $E_{ij} = (e_{mn})$ s.t. $e_{mn} = \begin{cases} 1 & \text{if } (m,n) = (i,j) \\ 0 & \text{else} \end{cases}$ for $i \neq j$ "almost permute" these vectors, i.e. $E_{ij} \cdot \mathbf{v_j} = \mathbf{v_i}$ and $E_{ij} \cdot \mathbf{v_k} = \mathbf{0}$ for $k \neq j$.
3. In fact we only need to use the matrices $F_i = E_{i+1\ i}$ to reach any basis vector from $\mathbf{v_1}$.

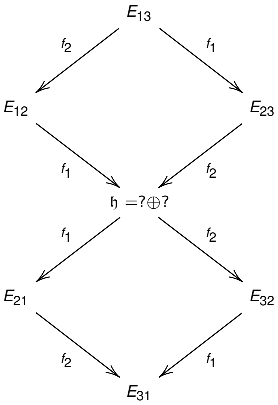
We say that this is a **good basis**.

Thus we can encode the representation as a colored directed graph, for example, $\mathfrak{sl}_3$ acting on $\mathbb{C}^3$ could be represented like this:

$$\mathbf{v_1} \xrightarrow{f_1} \mathbf{v_2} \xrightarrow{f_2} \mathbf{v_3}$$

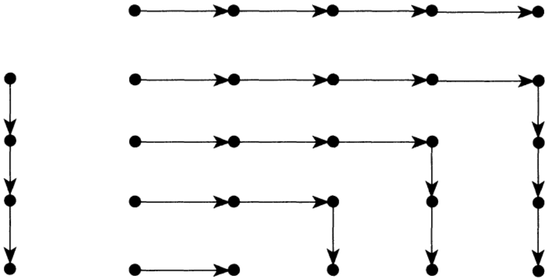
Our aim is to generalize this idea and we'd hope that the good basis is compatible with tensor product decompositions and branching.

This works splendidly only as long as each weight space is one-dimensional. We already run into trouble with the adjoint representation of $\mathfrak{sl}_3$, as $\ker f_1, \ker f_2, \operatorname{im} f_1, \operatorname{im} f_2$ are all different subspaces of $\mathfrak{h}$.

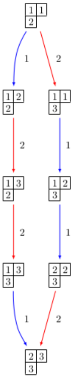


Fortunately there is a way of fixing this problem, due to Kashiwara [6] , by going first to $U_q(\mathfrak{g})$ and then taking a limit as $q \to 0$ in a suitable sense. It turns out that in this setting, choosing a good basis is always possible. This object, the colored directed graph consisting of the basis elements and the action of the f_is is called a **crystal**.

What is the benefit of crystals? Combinatorics. For $\mathfrak{g} = \mathfrak{sl}_2$-crystals, tensor product decompositions are given by:



We know that for an irreducible $\mathfrak{sl}_n$-representation V_λ of highest weight λ, $\dim(V_\lambda) = \#SSYT(\lambda)$ with entries up to n. The crystal of V_λ has a realization as tableaux, for example, here is the crystal of the adjoint representation of $\mathfrak{sl}_3$



The lowering operators f_i work as follows:

1. Read the entries of the tableau from bottom to top, left to right, ignoring all numbers except i and $i+1$.
2. Replace $i+1 \to$ (and $i \to$),
3. Turn the i corresponding to the rightmost unmatched) into an $i+1$

Example: applying f_2

1	2	2	4
2	3	4	
4	4		

$\to$ 442341224 $\to$)()) $\to$ 2323 $\to$

1	2	3	4
2	3	4	
4	4		

Now consider the affine Lie algebra $\tilde{\mathfrak{g}}$. This is a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, we may consider $U(\tilde{\mathfrak{g}})$ and $U_q(\tilde{\mathfrak{g}})$.

Conjecture 1 (Hatayama et al. [5])

Certain finite-dimensional $U(\tilde{\mathfrak{g}})$-modules, called ***Kirillov-Reshetikhin modules*** *have crystal bases. A KR-module $W_s^{(r)}$ is determined by a choice of a non-affine node (r) of the Dynkin diagram and a positive integer s.*

As before, the situation is simplest for $\widetilde{\mathfrak{sl}_n}$. It turns out that in this case, each KR-module stays irreducible when restricted to $\mathfrak{sl}_n$, for example, for $U_q(\mathfrak{sl}_3)$, W_1^1 is the standard representation, and the KR crystal is

Question: can we extend the tableau model to the KR setting?
We want to give a combinatorial description of the f_0 operator.
Shimozono's [9] method for $\mathfrak{sl}_n$: use Schützenberger's **promotion** operator on tableaux.
If T is an SSYT, then $pr(T)$ is obtained by the following procedure:

1. Erase all entries n from the tableau.
2. Jeu-de-taquin the other boxes to the Southeast.
3. Fill the empty boxes in the Northwest with zeros.
4. Add one to each entry.

1	2	2	4
2	3	4	
4	4		

erase 4 $\rightarrow$

1	2	2	
2	3		

jdt $\rightarrow$

1	2	2	
2	3		

fill & increment $\rightarrow$

1	1	1	1
2	3	3	
3	4		

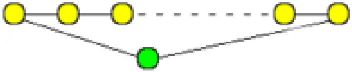
The promotion operator shifts the content of the tableau cyclically, and it is in fact cyclic of order n for rectangular tableaux, but not in general. Notice how pr interacts with the lowering operators for rectangular tableaux:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{pr} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{f_2} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \xrightarrow{pr^{-1}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \xrightarrow{f_1} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

So the promotion operator realizes the cyclic symmetry of the affine type A Dynkin diagram, at least for rectangular tableaux.



Theorem 2 (Shimozono)

$$f_0 = pr^{-1} \circ f_1 \circ pr$$

One would think that this is a phenomenon specific to type A because of the cyclic symmetry, but

Theorem 3 (Fourier, Okado, Schilling, [4])

There exist tableaux models for other types, where the KR crystals were shown to exist on a case-by-case basis by finding a suitable analogue of the promotion operator

The **cactus group** $J_{\mathfrak{g}}$ was introduced by Henriques and Kamnitzer [11] in the context of coboundary categories. It has

1. Generators: s_J for J a connected subdiagram of $\mathfrak{g}$'s Dynkin diagram.
2. Relations:
   1. $s_J^2 = 1 \; \forall J$.
   2. $s_J s_{J'} = s_{J'} s_J$ if $J \cup J'$ is not connected.
   3. $s_J s_{J'} = s_{\theta_J(J')} s_J$ for $J' \subset J$.

   where θ_J is the Dynkin diagram automorphism $-w_0^J$ of J.

It surjects onto $W_{\mathfrak{g}}$ by $s_J \mapsto w_0^J$.

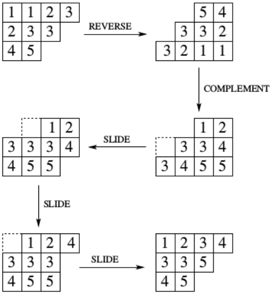
Halacheva [10] showed that there is an action of $J_{\mathfrak{g}}$ on any $\mathfrak{g}$-crystal, which we now describe.
The **Schützenberger involution** on a crystal B_λ of an irrep V_λ is the unique map $\xi_\lambda : B_\lambda \to B_\lambda$ on the vertices satisfying

1. $e_i(\xi_\lambda(b)) = \xi_\lambda(f_{\theta(i)}(b))$
2. $f_i(\xi_\lambda(b)) = \xi_\lambda(e_{\theta(i)}(b))$
3. $wt(\xi_\lambda(b)) = w_0 \cdot (wt(b))$

So, in effect, it flips the crystal upside down.

For $\mathfrak{sl}_n$-crystals, the operation is given by **evacuation** on tableaux, which is the following procedure:

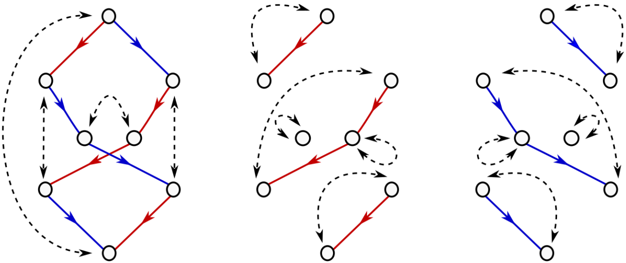
1. Rotate the tableau 180°.
2. Complement the entries $i \to n + 1 - i$.
3. Jeu-de-taquin the tableau back to the original shape.



Halacheva [10] showed that $J_{\mathfrak{g}}$ acts on a $\mathfrak{g}$-crystal B (as a set) by

$$s_J(b) = \xi_{B_J}(b)$$

where ξ_{B_J} is the Schützenberger involution on the restricted crystal using only the lowering operators in J. For example, on the adjoint representation of $\mathfrak{sl}_3$,



The s_{12} action. The s_1 action. The s_2 action.

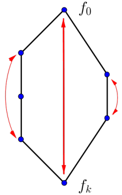
Theorem 4 (Kwon, [7])

For a KR-module corresponding to a cominuscule fundamental weight ω_k, let J be the subset of $\mathfrak{g}$'s Dynkin diagram complementary to a (possibly different) cominuscule weight ω_j. Then

$$f_0^{-1} = e_0 = s_J f_j s_J.$$

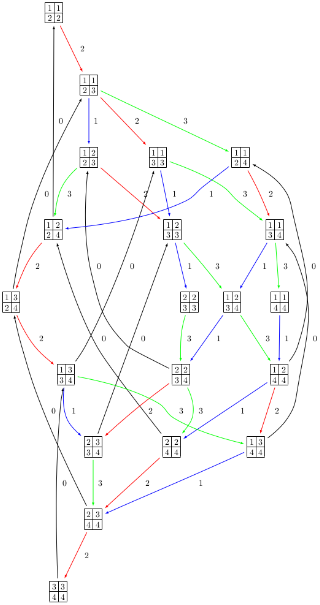
Kwon deduces this as a corollary of his main result, which is exploiting the representation-theoretic consequences of the Robinson-Schensted-Knuth correspondence. His proof is not cactus-theoretic, and is still case by case, even though the statement is uniform.

In type A, s_J corresponds to



Remark 5

Note that we needed to restrict our attention to a KR-module associated to a cominuscule fundamental weight. This is equivalent to the KR-module being irreducible as a $U_q(\mathfrak{g})$-module (not just as a $U_q(\widetilde{\mathfrak{g}})$-module). The cactus action preserves the $U_q(\mathfrak{g})$-connected components of the crystals, so there is no hope in using it in KR-modules associated to non-cominuscule weights.


1 1
2 2
1 1
2 3
1 2
2 3
1 1
3 3
1 1
2 4
1 2
2 4
1 2
3 3
1 1
3 4
1 3
2 4
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4 4

It would be satisfying to have a cactus-theoretic uniform proof of Kwon's theorem.

Definition 6

The i-th **Bender-Knuth move** t_i acts on a tableau, changing some is to $i+1$s and vice versa. In each row, it switches the number of is that have no $i+1$s directly underneath them with the number of $i+1$s that have no is directly above them.

Example 7

$$t_2\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&3&4\\\cline{1-3} 4&4\\\cline{1-2}\end{array}\right)=\begin{array}{|c|c|c|c|}\hline 1&2&2&4\\\hline 3&3&4\\\cline{1-3} 4&4\\\cline{1-2}\end{array}$$

In [2], Chmutov, Glick and Pylyavskyy show that the Cactus group surjects onto the BK-group, and the action of the Cactus group on crystals factors through the BK group. The relations satisfied by the BK moves turn out to be more manageable than those of the natural generators of the Cactus group, and with some combinatorics, one can reprove Kwon's result in type A using this map.

Remark 8

It is hard to give a type-independent proof if the group one is using is specific to type A, so the question is: is there an analogue of the BK group in other types?

There is a well-known bijection between semistandard Young tableaux and Gelfand-Tsetlin patterns. In type A, for representations $V_{l\omega_k}$, this can be used to give a model for a crystal that consists of poset maps $\{f : \{\text{roots above } \alpha_k\} \to [l]\}$, and the same set gives a model for the crystal in other types as long as ω_k is *minuscule*. The BK moves have a nice description in terms of this model, consisting of so-called **toggles**, and these can be defined on any poset. The hope is that the type A argument can be generalized to at least the simply-laced types, where minuscule and cominuscule weights coincide.

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