Vanishing of Littlewood-Richardson polynomials is in P

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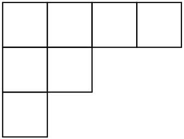
Background

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with n nonnegative parts.

Definition

The **initial diagram** for λ is a grid with n rows and $m \geq n + \lambda_1 - 1$ columns in which the Young diagram of λ is in the northwest corner.

For example, $\lambda = (4, 2, 1, 0)$



$$\longleftrightarrow \begin{pmatrix} + & + & + & + & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Background

Definition

A **local move** is a mutation of any 2×2 subsquare of the form

$$\begin{matrix} + & \cdot \\ \cdot & \cdot \end{matrix} \quad \mapsto \quad \begin{matrix} \cdot & \cdot \\ \cdot & + \end{matrix}$$

The configurations of $+$'s in the grid resulting from local moves on the initial diagram for λ are called **plus diagrams**.

For example,

$$\begin{pmatrix} + & + & + & \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \begin{pmatrix} + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot & \cdot & + & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Background

To a diagram T, we can assign weights,

$$wt_x(T) = \Pi_i x_i^{+\text{'s in row } i}$$

Or finer,

$$wt_{x,y}(T) = \Pi_{(i,j)\in T}(x_i - y_j)$$

For example, for the following $T \in \text{Plus}((3,2,0))$,

$$\begin{pmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & + & \cdot & + \end{pmatrix}$$

$$wt_x(T) = x_1 x_2^2 x_3^2$$

$$wt_{x,y}(T) = (x_1 - y_1)(x_2 - y_1)(x_2 - y_3)(x_3 - y_3)(x_3 - y_5).$$

Background

We can realize the Schur function as

$$s_\lambda(X) = \sum_{Plus(\lambda)} wt_x(T)$$

which forms a $\mathbb{Z}$-linear basis of $Sym[X]$ and similarly the factorial Schur function as

$$s_\lambda(X;Y) = \sum_{Plus(\lambda)} wt_{x,y}(T)$$

which forms a $\mathbb{Z}[Y]$-linear basis of $Sym[X] \otimes_{\mathbb{Z}} \mathbb{Z}[Y]$.

Structure Coefficients

Then we can discuss their structure coefficients:

$$s_\lambda(X)s_\mu(X) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(X)$$

where $c^\nu_{\lambda,\mu} \geq 0$ is called the **Littlewood-Richardson coefficient**. Extending this,

$$s_\lambda(X;Y)s_\mu(X;Y) = \sum_\nu C^\nu_{\lambda,\mu}(Y) s_\nu(X;Y)$$

where $C^\nu_{\lambda,\mu}$ is called the **Littlewood-Richardson polynomial**. For example, using $\lambda = \mu = (1,0)$ we can express

$$\begin{aligned} s_{(1,0)}(x_1,x_2;Y)^2 &= s_{(2,0)}(x_1,x_2;Y) + s_{(1,1)}(x_1,x_2;Y) \\ &\quad + (y_3 - y_2)s_{(1,0)}(x_1,x_2;Y). \end{aligned}$$

Complexity of $c_{\lambda,\mu}^{\nu}$

Theorem (DeLoera, McAllister, 2006; Mulmuley, Narayanan, Sohoni, 2012)

$c_{\lambda,\mu}^{\nu} \neq 0$ *can be decided in polynomial time.*

Theorem (Adve, Robichaux, Yong, 2017)

$C_{\lambda,\mu}^{\nu}(Y) \not\equiv 0$ *can be decided in polynomial time.*

Eigenvalue Problem

Problem

For A, B, C $r \times r$ Hermitian matrices with real eigenvalues λ, μ, ν, how does $A + B = C$ constrain $(\lambda, \mu, \nu) \in (\mathbb{R}^r)^3$?

Horn (1962) conjectured a recursive list of inequalities, and Klyachko (1998) proved another list of inequalities. Klyachko's inequalities are satisfied by $(\lambda, \mu, \nu) \in \mathbb{Z}^{3r}_{\geq 0} \iff c^{N\nu}_{N\lambda, N\mu} \neq 0$ for some N, which imply those of Horn given another result.

Theorem (Knutson-Tao, 1999)

$c^{\nu}_{\lambda,\mu} \neq 0 \iff c^{N\nu}_{N\lambda, N\mu} \neq 0$ *for all* $N \in \mathbb{Z}_{>0}$.

These inequalities precisely control the vanishing of $c^{\nu}_{\lambda,\mu}$.

A Variation on the Eigenvalue Problem

Problem

For A, B, C Hermitian matrices with real eigenvalues λ, μ, ν, how does $A + B \geq C$ constrain $(\lambda, \mu, \nu) \in (\mathbb{R}^r)^3$?

Friedland (2000) proved inequalities that constrain those eigenvalues.

Theorem (Anderson-Richmond-Yong, 2013)

$C^{\nu}_{\lambda,\mu} \neq 0 \iff C^{N\nu}_{N\lambda,N\mu} \neq 0$ *for all* $N \in \mathbb{Z}_{>0}$.

Combining this result with Friedland's, they show that Friedland's inequalities control the vanishing of $C^{\nu}_{\lambda,\mu}$.

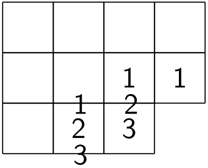
Proving the Complexity of $C^{\nu}_{\lambda,\mu}$

Our proof is a modification of the argument of Mulmuley, Narayanan, Sohoni.

We use a combinatorial rule for $C^{\nu}_{\lambda,\mu}$ in terms of edge-labeled tableaux of Thomas and Yong to construct a polytope $P^{\nu}_{\lambda,\mu}$ where

- $P^{\nu}_{\lambda,\mu} \cap \mathbb{Z}^{2\ell(\nu)\ell(\mu)} \neq \varnothing$ if and only if $C^{\nu}_{\lambda,\mu} > 0$, and
- $NP^{\nu}_{\lambda,\mu} = P^{N\nu}_{N\lambda,N\mu}$.

For example, T is an edge-labeled tableau that would be detected by $C^{(3,2,2)}_{(4,4,3),(4,2,1)}$ is

$T =$ 

Proof Sketch

- $P^{\nu}_{\lambda,\mu} \neq \varnothing \iff C^{\nu}_{\lambda,\mu} \not\equiv 0$.
  ($\Rightarrow$) $P^{\nu}_{\lambda,\mu} \neq \varnothing$ implies there exists a rational vertex.
  Then for some N, $NP^{\nu}_{\lambda,\mu} = P^{N\nu}_{N\lambda,N\mu}$ has a lattice point.
  By construction of $P^{N\nu}_{N\lambda,N\mu}$, $C^{N\nu}_{N\lambda,N\mu} \not\equiv 0$.
  By equivariant saturation, $C^{\nu}_{\lambda,\mu} \not\equiv 0$.

- Considering the linear programming problem and any objective function, we need to know the feasibility.
  - Simplex method has worst case exponential complexity.
  - Ellipsoid method has polynomial time complexity.

Conclusion and Summary

- The factorial schur functions $s_\lambda(X;Y)$ enrich the ordinary schur functions $s_\lambda(X)$.
- Through a variation on the eigenvalue problem, we discussed a connection between eigenvalues λ, μ, ν of matrices $A + B \geq C$ and when $C^{\nu}_{\lambda,\mu} \not\equiv 0$.
- We proved $C^{\nu}_{\lambda,\mu} \not\equiv 0$ is polynomial time to decide by combining combinatorial rule with equivariant saturation and integer programming.