Schur algebras and quantum symmetric pairs with unequal parameters

Chun-Ju Lai

University of Georgia

cjlai@uga.edu

Oct 19-21, 2018

(joint work with L. Luo)

Schur duality

- $\mathfrak{gl}_n =$ general linear Lie algebra $\curvearrowright V = \mathbb{C}^n$: natural representation
- $\Sigma_d =$ symmetric group on d letters
- Schur duality $=$ double centralizer property:

$$\begin{tikzcd}
\underset{\text{Universal enveloping algebra}}{U(\mathfrak{gl}_n)} \arrow[d, two heads] & & \\
\underset{\text{Schur algebra}}{S(n, d)} \arrow[r, phantom, "\curvearrowright"] & \underset{\text{tensor space}}{V^{\otimes d}} \arrow[r, phantom, "\curvearrowleft"] & \underset{\text{group algebra}}{\mathbb{C}(\Sigma_d)}
\end{tikzcd}$$

- Schur algebra $S(n, d) = \mathrm{End}_{\mathbb{C}(\Sigma_d)}(V^{\otimes d})$

Quantization

- [Jimbo'86] q-Schur duality

$$\begin{tikzcd}
\underset{\text{univ env algebra}}{U(\mathfrak{gl}_n)} \arrow[r, phantom, "\curvearrowright"] \arrow[d, dotted, "\text{quantization}"'] & (\mathbb{C}^n)^{\otimes d} & \underset{\text{group algebra}}{\mathbb{C}[\Sigma_d]} \arrow[l, phantom, "\curvearrowleft"] \arrow[d, dotted, "\text{quantization}"] \\
\underset{\text{quantum group}}{\mathbf{U}_q(\mathfrak{gl}_n)} \arrow[r, phantom, "\curvearrowright"] & (\mathbb{Q}(q)^n)^{\otimes d} & \underset{\text{Hecke algebra}}{\mathbf{H}_q(\Sigma_d)} \arrow[l, phantom, "\curvearrowleft"]
\end{tikzcd}$$

For convenience our ground fields are $\mathbb{Q}(v)$ with $v = q^{1/2}$ an indeterminate

q-Schur algebra

- Double centralizer property

$$\begin{tikzcd}
\underset{\text{quantum group}}{\mathbf{U}_q(\mathfrak{gl}_n)} \arrow[d, two heads] & & & & \\
\underset{\text{q-Schur algebra}}{\mathbf{S}_q(n, d)} & \curvearrowright & \underset{\text{tensor space}}{\mathbf{V}_q^{\otimes d}} & \curvearrowleft & \underset{\text{Hecke algebra}}{\mathbf{H}_q(\Sigma_d)}
\end{tikzcd}$$

- q-Schur algebra $\mathbf{S}_q(n, d) = \mathrm{End}_{\mathbf{H}_q(\Sigma_d)}(\mathbf{V}_q^{\otimes d})$
  - $\mathbf{S}_q(n, d)$ has canonical basis $\leftrightarrow$ Kazhdan-Lusztig basis of $\mathbf{H}_q(\Sigma_d)$
  - $\mathbf{S}_q(n, d)$ is a quotient of quantum group
- "top-bottom approach"

Quantum group arise from Schur duality

- [Beilinson-Lusztig-MacPherson '90]
  "Bottom-top approach"

stabilization
n fixed, $d \in \mathbb{N}$

$$\begin{tikzcd}
{}^{S}\dot{\mathbf{U}}_n & & \\
\mathbf{S}_q(n,d) \arrow[u, Rightarrow] \arrow[r, phantom, "\curvearrowright"] & \mathbf{V}_q^{\otimes d} & \mathbf{H}_q(\Sigma_d) \arrow[l, phantom, "\curvearrowleft"]
\end{tikzcd}$$

q-Schur algebra

Hecke algebra

- Canonical bases of $\mathbf{S}_q(n,d)$, $d \in \mathbb{N}$, lift to canonical basis of ${}^{S}\dot{\mathbf{U}}_n$
- ${}^{S}\dot{\mathbf{U}}_n \simeq \dot{\mathbf{U}}_q(\mathfrak{gl}_n)$: modified quantum $\mathfrak{gl}_n$

$\Rightarrow$ A concrete realization of canonical basis of $\dot{\mathbf{U}}_q(\mathfrak{gl}_n)$

Geometric Schur duality

- $G = \mathrm{GL}_d(\mathbb{F}_q) \curvearrowright \mathbb{F}_q^d$ and hence acts on
  - complete flags $\mathcal{Y} = \{ V = (\{0\} = V_0 \overset{1}{\subset} V_1 \overset{1}{\subset} \ldots \overset{1}{\subset} V_d = \mathbb{F}_q^d) \}$
  - n-step partial flags $\mathcal{X} = \{ V = (\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = \mathbb{F}_q^d) \}$
- Convolution algebras for $\mathcal{F}_i = \mathcal{X}$ or $\mathcal{Y}$:

$$\mathcal{A}_G(\mathcal{F}_1 \times \mathcal{F}_2) = \{ G\text{-invariant } f \colon \mathcal{F}_1 \times \mathcal{F}_2 \to \mathbb{Z}[v, v^{-1}] \}$$

- Geometric Schur duality

$$\begin{tikzcd}
\underset{q\text{-Schur algebra}}{\mathbf{S}_q(n,d)} \arrow[r, phantom, "\curvearrowright"] \arrow[d, equal] & \underset{\text{tensor space}}{\mathbf{V}_q^{\otimes d}} \arrow[r, phantom, "\curvearrowleft"] \arrow[d, equal] & \underset{\text{Hecke algebra}}{\mathbf{H}_q(\Sigma_d)} \arrow[d, equal] \\
\mathcal{A}_G(\mathcal{X} \times \mathcal{X}) \arrow[r, phantom, "\curvearrowright"] & \mathcal{A}_G(\mathcal{X} \times \mathcal{Y}) \arrow[r, phantom, "\curvearrowleft"] & \mathcal{A}_G(\mathcal{Y} \times \mathcal{Y})
\end{tikzcd}$$

- [BLM] relies on a dimension counting on $\mathcal{A}_G(\mathcal{X} \times \mathcal{X})$

Potential generalizations

- Starting with a family of modules $M = M_{n,d}$ of type X Hecke algebra, we can define its centralizing partner

$$\mathbf{S}^{\mathsf{X}}_{n,d} := \mathrm{End}_{\mathbf{H}^{\mathsf{X}}_d}(M) \quad \curvearrowright \quad M \quad \curvearrowleft \quad \underset{\text{type X Hecke algebra}}{\mathbf{H}^{\mathsf{X}}_d}$$

- Questions:
  ① Does $\mathbf{S}^{\mathsf{X}}_{n,d}$ have nice bases?
  ② When does double centralizer property hold? i.e., $\mathrm{End}_{\mathbf{S}^{\mathsf{X}}_{n,d}}(M) = \mathbf{H}^{\mathsf{X}}_d$?
  ③ Does the BLM construction apply? i.e., ${}^{S}\dot{\mathbf{U}}(M) := \underset{\longleftarrow}{\mathrm{Stab}}\, \mathbf{S}^{\mathsf{X}}_{n,d}$ exists and has compatible nice bases?
  ④ If so, what is the algebra ${}^{S}\dot{\mathbf{U}}(M)$?

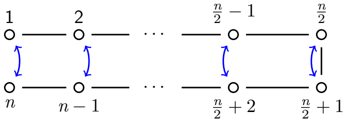
Type B

- [Dipper-James-Mathas'98] The cyclotomic Hecke algebra $\mathbf{H}_{Q,u_1,\dots,u_r}(r,1,n) = \mathbf{H}_q^{\mathsf{B}}(d)$ at certain specialization.
If $M =$ certain q-permutation module, then
  1. Cyclotomic Schur algebra has cellular basis
  2. Double centralizer property is unclear in general
  3. BLM construction is unclear
- [Bao-Kujawa-Li-Wang'14] If $M = \mathcal{A}_{O(2d+1)}(\mathcal{X} \times \mathcal{Y})$ for some type B flags, then
  1. Type B q-Schur algebra has canonical basis
  2. Double centralizer property holds when $n \geq d$
  3. Stabilization procedure applies and the canonical bases are compatible
  4. ${}^{S}\dot{\mathbf{U}}(M)$ is a (idempotented) coideal subalgebra in QSP of type A III/IV

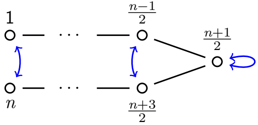
An algebraic approach is available for $M =$ a different q-permutation module

“Involutive quantum groups”

- Satake diagrams of type A III/IV (i.e., Dynkin diagrams of type A with involutions $\theta = \imath, \jmath$):



$\theta = \jmath : n$ even



$\theta = \imath : n$ odd

- classical symmetric pair $(\mathfrak{gl}_n, \mathfrak{gl}_n^\theta) \xrightarrow{\text{quantize}}$ quantum symmetric pair $(\mathbf{U}_q(\mathfrak{gl}_n), \mathbf{U}_n^\theta)$
- ${}^S\dot{\mathbf{U}}(M) =$ (idempotented) $\mathbf{U}_n^\imath$ or $\mathbf{U}_n^\jmath$

Applications

- [Bao-Wang'13] The canonical basis theory for $\mathbf{U}^{\imath}_n, \mathbf{U}^{\jmath}_n$
  - gives a new formulation of KL theory for Lie algebra of type B/C
  - establishes KL theory for category $\mathcal{O}$ of Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ (i.e. super type B/C)
- [Bao'16] Reformulation of KL theory for category $\mathcal{O}$ of Lie algebra of type D and for Lie superalgebra $\mathfrak{osp}(2m|2n)$ (i.e. super type D) via
  - An multiparameter upgrade $\mathbb{U}^{\imath}_n, \mathbb{U}^{\jmath}_n$ of QSP over $\mathbb{Q}(p, q)$
  - A specialization at $p = 1$

⚠ The BLM construction for $\mathbb{U}^{\imath}_n, \mathbb{U}^{\jmath}_n$ was unknown.

Obstacles in such BLM construction

1. A geometric approach (i.e., counting over finite fields) for two parameters is not known

⇒ We use a algebraic/combinatorial approach via Hecke algebras of type B over $\mathbb{Q}(p, q)$

2. A canonical basis theory over $\mathbb{Q}(p, q)$ is not known

⇒ We generalize Lusztig's theory for Hecke algebras with unequal parameters to Schur algebras with unequal parameters, and use it to establish a canonical basis theory at specialization

Hecke algebras with unequal parameters

- Let $p = u^2, q = v^2$. The multiparameter Hecke algebra $\mathbb{H}$ of Weyl group $W = W_d^{\mathrm{B/C}}$ is an $\mathbb{Z}[u^{\pm}, v^{\pm}]$-algebra with basis $\{T_w \mid w \in W\}$ satisfying

$$\begin{aligned} T_w T_{w'} &= T_{ww'} && \text{if} \quad \ell(ww') = \ell(w) + \ell(w'), \\ (T_{s_0} + 1)(T_{s_0} - p) &= 0, \\ (T_{s_i} + 1)(T_{s_i} - q) &= 0 && \text{for} \quad 1 \leq i \leq d-1. \end{aligned}$$

- Bar involution $\overline{} : \mathbb{H} \to \mathbb{H}$ by $u \mapsto u^{-1}, v \mapsto v^{-1}$

Hecke algebras with unequal parameters

- A weight function $\mathbf{L}: W \to \mathbb{N}$ is a map satisfying

$$\mathbf{L}(ww') = \mathbf{L}(w) + \mathbf{L}(w') \quad \text{for } \substack{w, w' \in W \text{ such that} \\ \ell(ww') = \ell(w) + \ell(w')}$$

- $\mathbf{L} = \ell$ if $\mathbf{L}(s) = 1$.
- Lusztig showed that, for any weight function $\mathbf{L}$, there is a bar-invariant basis $\{C_w^{\mathbf{L}}\}$ at specialization $u = \mathbf{v}^{\mathbf{L}(s_0)}, v = \mathbf{v}^{\mathbf{L}(s_1)}$, given by

$$C_w^{\mathbf{L}} = u^{\mathsf{pwr}} v^{\mathsf{pwr}} \sum_{y \leq w} p_{y,w}(\mathbf{v})\, T_y,$$

where $p_{y,w}(\mathbf{v})$ is an analogue of Kazhdan-Lusztig polynomial.

Schur algebras with unequal parameters

- The Schur algebra $\mathbb{S}_{n,d} = \mathrm{End}_{\mathbb{H}}\Big(\bigoplus_{\lambda\in\Lambda} x_\lambda\mathbb{H}\Big) = \bigoplus_{\lambda,\mu\in\Lambda} \mathrm{Hom}_{\mathbb{H}}(x_\mu\mathbb{H}, x_\lambda\mathbb{H})$, where $\bigoplus_{\lambda\in\Lambda} x_\lambda\mathbb{H}$ is a deformation of permutation modules.
- $\mathbb{S}_{n,d}$ has a Dipper-James basis $\{e_A\}$ characterized by combinatorics of Hecke algebras.
- Bar involution $\overline{} : \mathbb{S}_{n,d} \to \mathbb{S}_{n,d}$ by $\bar{f}(x_\mu) = \overline{f(\overline{x_\mu})}$

Theorem 1 (Lai-Luo'18)

1. As an algebra, $\mathbb{S}_{n,d}$ is generated by $\{e_B \mid B \text{ is Chevalley}\}$.
2. The structral constants with Chevalley generators are computed.

Schur algebras with unequal parameters

- Missing canonical basis theory of $\mathbb{S}_{n,d}$ since
  1. lack of standard basis $\{[A]\}$ satisfying an integral unitriangular condition:
  $$\overline{[A]} \in [A] + \sum_{B<A} \mathbb{Z}[u^{\pm}, v^{\pm}][B].$$
  2. lack of multiparameter KL polynomials
- Using Lusztig's basis $\{C_w^{\mathbf{L}}\}$ at the specialization, we obtain a canonical basis theory for $\mathbb{S}_{n,d}$ at specialization.

Theorem 2 (Lai-Luo'18)

There exists a monomial basis $\{m_A\}$ for $\mathbb{S}_{n,d}$.
$\Rightarrow$ there exists a canonical basis $\{\{A\}^{\mathbf{L}}\}$ for $\mathbb{S}_{n,d}$ at a specialization associated to a weight function $\mathbf{L}$.

Schur dualities

- Type B specialization

$$\begin{tikzcd}
\mathbb{S}^{\mathsf{B/C}}_{n,d} \curvearrowright \mathbb{V}^{\otimes d} \curvearrowleft \mathbb{H}^{\mathsf{B/C}}_{d} \arrow[d, Rightarrow, "\text{specialization at } u = \mathbf{v}^{\mathbf{L}(s_0)},\, v = \mathbf{v}^{\mathbf{L}(s_1)}"] & \text{over } \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \\
\mathbb{S}^{\mathbf{L},\mathsf{B/C}}_{n,d} \curvearrowright \mathbb{V}^{\otimes d}_{\mathbf{L}} \curvearrowleft \mathbb{H}^{\mathbf{L},\mathsf{B/C}}_{d} \arrow[d, Rightarrow, "\text{specialization at } u = v = \mathbf{v}"] & \text{over } \mathbb{Z}[\mathbf{v}^{\pm \mathbf{c}}] \\
\mathbf{S}^{\mathsf{B/C}}_{n,d} \curvearrowright \mathbf{V}^{\otimes d} \curvearrowleft \mathbf{H}^{\mathsf{B/C}}_{d} & \text{over } \mathbb{Z}[v, v^{-1}]
\end{tikzcd}$$

- At the specialization $u = 1, v = \mathbf{v}$, the quantum group arise from such Schur duality is used in [Bao'16] to formulate the KL theory of super type D.

Stablization

- We further establish a multiparameter stabilization for the first time

$$\begin{tikzcd}
\begin{matrix} {}^{S}\dot{\mathbb{U}}^{\mathrm{B/C}}_{n} \\ \text{stabilization algebra} \end{matrix} & & \\
\begin{matrix} \mathbb{S}_{n,d} \\ \text{Schur algebra} \end{matrix} \arrow[u, Rightarrow] \arrow[r, phantom, "\curvearrowright"] & \mathbb{V}^{\otimes d} & \begin{matrix} \mathbb{H}^{\mathrm{B/C}}_{d} \\ \text{Hecke algebra} \end{matrix} \arrow[l, phantom, "\curvearrowleft"]
\end{tikzcd}$$

Theorem 3 (Lai-Luo'18)

There exists a monomial basis $\{m_A\}$ for ${}^{S}\dot{\mathbb{U}}^{\mathrm{B/C}}_{n}$
$\Rightarrow$ there exists a canonical basis $\{\{A\}^{\mathbf{L}}\}$ for ${}^{S}\dot{\mathbb{U}}^{\mathrm{B/C}}_{n}$ at a specialization associated to a weight function $\mathbf{L}$.

Theorem 4 (Lai-Luo'18)

${}^{S}\dot{\mathbb{U}}^{\mathrm{B/C}}_{n}$ is isomorphic to $\mathbb{U}^{\imath}_{n}$ (or $\mathbb{U}^{\jmath}_{n}$, depending on parity)

Future direction

1. Multiparameter BLM constructions:
   - Cyclotomic Schur algebras $\Rightarrow$?
   - Multiparameter affine Schur algebras of type C $\overset{?}{\Rightarrow}$ multiparameter affine QSP
2. BLM construction for modules M for other algebras
3. Lifting other bases such as the cellular bases

Thank you for your attention