Double Affine Bruhat Order

Mandy Welch

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Notation

- Φ_{fin} - simply laced, irreducible root system
- Δ_{fin} - the set of simple roots
- W_{fin} - finite Weyl group
- Q - the root lattice
- $W_{\text{aff}} = Q \rtimes W_{\text{fin}}$ - the affine Weyl group
- Λ_i for $i = 0, 1, 2, \ldots, n$ - the affine fundamental weights

We have a pairing $\langle \ , \ \rangle$ such that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi_{\text{fin}}$. This allows us to identify Φ_{fin} with $\Phi_{\text{fin}}^{\vee}$, the set of coroots.

Motivation

We create the affine Weyl group W_{aff} by taking the semidirect product of the translation group associated to Q with W_{fin}.

$$W_{\mathrm{aff}} = Q \rtimes W_{\mathrm{fin}} = \{Y^{\lambda}w : \lambda \in Q, w \in W_{\mathrm{fin}}\}$$

Both W_{aff} and W_{fin} are Coxeter Groups. What happens if we take the affine Weyl group and try to do something similar?

The Tits Cone

Define $Q_{\mathrm{aff}} = Q \oplus \mathbb{Z}\delta$ and $X = Q \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$.

Extend the symmetric bilinear form $\langle Q, Q\rangle \to \mathbb{Z}$ to $\langle X, Q_{\mathrm{aff}}\rangle \to \mathbb{Z}$ by

$$\langle \delta, \delta\rangle = \langle Q, \delta\rangle = \langle \delta, Q\rangle = \langle \Lambda_0,\ Q\rangle = 0,\ \langle \Lambda_0, \delta\rangle = 1.$$

Let X_{dom} be the set of all dominant elements of X. Then the Tits cone $\mathcal{T}$ is given by

$$\mathcal{T} = \cup_{w \in W_{\mathrm{aff}}} w(X_{\mathrm{dom}}).$$

$$\mathcal{T} = \{m\delta : m \in \mathbb{Z}\} \cup \{\mu + m\delta + l\Lambda_0 : \mu \in Q, m \in \mathbb{Z}, l \in \mathbb{Z}_+\}$$

Double Affine Weyl Semigroup

We define the double affine Weyl semigroup W to be the semidirect product of the the translation semigroup associated to $\mathcal{T}$ with W_{aff}.

$$\begin{aligned} W =& \mathcal{T} \rtimes W_{\mathrm{aff}} \\ =& \{X^{\zeta}\tilde{w} : \zeta \in \mathcal{T}, \tilde{w} \in W_{\mathrm{aff}}\} \\ =& \{X^{\zeta}Y^{\lambda}w : \zeta \in \mathcal{T}, \lambda \in Q, w \in W_{\mathrm{fin}}\} \end{aligned}$$

This is a semigroup, as it is not closed under inverses.

Double Affine Roots

The set of double affine roots is given by

$$\Phi = \{\tilde{\alpha} + j\pi : \tilde{\alpha} \in \Phi_{\mathrm{aff}}, j \in \mathbb{Z}\} = \{\nu + r\delta + j\pi : \nu \in \Phi_{\mathrm{fin}}, r, j \in \mathbb{Z}\}.$$

We say that a double affine root $\alpha = \tilde{\alpha} + j\pi$ is positive if $\tilde{\alpha} > 0$ and $j \geq 0$, or $\tilde{\alpha} < 0$ and $j > 0$.

W acts on Φ by

$$X^{\zeta}\tilde{w}(\tilde{\alpha} + j\pi) = \tilde{w}(\tilde{\alpha}) + (j - \langle \zeta, \tilde{w}(\tilde{\alpha}) \rangle)\pi$$

Bruhat Order

Given $x \in W$ and α a positive double affine root, [BKP] defined

$$x \geq s_\alpha x \iff x^{-1}(\alpha) < 0.$$

In [MO] it was shown

$$x \geq s_\alpha x \iff \ell(x) \geq \ell(s_\alpha x).$$

Cocovers

Let $x, y \in W$. Then y is said to be a cocover of x if $x > y$ and there is no $z \in W$ such that $x > z > y$.

In [MO] it was shown that for α a positive double affine root,

$$s_\alpha x \text{ is a cocover of } x \iff \ell(x) = \ell(s_\alpha x) + 1.$$

Setup for Cocover Conditions

1. $x = X^{\tilde{v}\zeta}\tilde{w} \in W$ where ζ is dominant
2. $y = s_\alpha x$
3. $\alpha = -\tilde{v}\tilde{\alpha} + j\pi$ is a positive double affine root
4. $\ell(\tilde{w}),\ \ell(s_{\tilde{v}\tilde{\alpha}}\tilde{w}) \leq M$
5. $\langle \zeta, \alpha_i \rangle \geq 2(M+1)$ for $i = 0, 1, 2, \ldots, n$

We wish to determine when y is a cocover of x.

Length Conditions

Theorem (Generalization of [LS] and [Mi])

With x and y as defined above, y is a cocover of x if and only if one of the following hold:

1. $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1$ *and* $j = 0$ *so* $y = X^{\tilde{v}s_{\tilde{\alpha}}\zeta} s_{\tilde{v}\tilde{\alpha}} \tilde{w}$
2. $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1 - \langle \tilde{\alpha}, 2\rho \rangle$ *and* $j = 1$ *so* $y = X^{\tilde{v}s_{\tilde{\alpha}}(\zeta - \tilde{\alpha})} s_{\tilde{v}\tilde{\alpha}} \tilde{w}$
3. $\ell(\tilde{w}^{-1}\tilde{v}s_{\tilde{\alpha}}) = \ell(\tilde{w}^{-1}\tilde{v}) + 1$ *and* $j = \langle \zeta, \tilde{\alpha} \rangle$ *so* $y = X^{\tilde{v}\zeta} s_{\tilde{v}\tilde{\alpha}} \tilde{w}$
4. $\ell(\tilde{w}^{-1}\tilde{v}s_{\tilde{\alpha}}) = \ell(\tilde{w}^{-1}\tilde{v}) + 1 - \langle \tilde{\alpha}, 2\rho \rangle$ *and* $j = \langle \zeta, \tilde{\alpha} \rangle - 1$ *so* $y = X^{\tilde{v}(\zeta - \tilde{\alpha})} s_{\tilde{v}\tilde{\alpha}} \tilde{w}$

Quantum Bruhat Graph of W_{aff}

- Vertices: Elements of W_{aff}
- Edges: For $\tilde{\alpha}$ positive, directed edge from $\tilde{v}s_{\tilde{\alpha}}$ to $\tilde{v}$ if one of the following hold:
  1. $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1$
  2. $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) - \langle \tilde{\alpha}, 2\rho \rangle + 1$

New Theorem

Setup:

1. $x = X^{\tilde{v}\zeta}\tilde{w}$
2. ζ is dominant and $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, 2, \ldots, n$.
3. $\tilde{\alpha}$ is an affine root such that $\alpha = -\tilde{v}(\tilde{\alpha}) + j\pi$ is positive

Theorem

With the setup given above, $y = s_\alpha x$ is a cocover of x if and only if one of the following holds:

1. $j = 0$ *and* $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1$.
2. $j = 1$ *and* $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1 - \langle \tilde{\alpha}, 2\rho \rangle$.
3. $j = \langle \zeta, \tilde{\alpha} \rangle$ *and* $\ell(\tilde{w}^{-1}\tilde{v}s_{\tilde{\alpha}}) = \ell(\tilde{w}^{-1}\tilde{v}) + 1$.
4. $j = \langle \zeta, \tilde{\alpha} \rangle - 1$ *and* $\ell(\tilde{w}^{-1}\tilde{v}s_{\tilde{\alpha}}) = \ell(\tilde{w}^{-1}\tilde{v}) + 1 - \langle \tilde{\alpha}, 2\rho \rangle$.

Length Difference Set

Theorem (MO)

Let $x = X^{\zeta}\tilde{w}$ with $\zeta \in \mathcal{T}$ and $\tilde{w} \in W_{\mathrm{aff}}$. Let α be a positive double affine root such that $x^{-1}(\alpha) < 0$. Then $y = s_\alpha x \leq x$ with respect to the Bruhat order and

$$\ell(y) = \ell(x) - |\{\beta > 0 \ : \ x^{-1}(\beta) < 0, \ \ s_\alpha(\beta) < 0, \ \ x^{-1}s_\alpha(\beta) > 0\}|$$

In particular, $L_{x,\alpha} = \{\beta > 0 \ : \ x^{-1}(\beta) < 0, \ \ s_\alpha(\beta) < 0, \ \ x^{-1}s_\alpha(\beta) > 0\}$ is finite.

Note: y is a cocover of x if and only if $L_{x,\alpha} = \{\alpha\}$.

Defining Graphs

Definition

Let $\Gamma_{x,\nu}$ denote the points $(r,j) \in \mathbb{Z}^2$ such that $\alpha = \nu + r\delta + j\pi > 0$ and $x^{-1}(\alpha) < 0$.

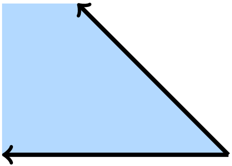

Figure: $\Gamma_{x,\nu}$

Corners

Notation: $\alpha = \nu + r\delta + j\pi$

Definition

For $\beta = \nu + p\delta + q\pi$, define β_α^- to be the root found by rotating (p, q) 180 degrees about (r, j).

Remark

If β and α have the same finite root, then $\beta_\alpha^- = -s_\alpha\beta$.

Definition

We say that α is a corner of the graph $\Gamma_{x,\nu}$ if α corresponds to a point in $\Gamma_{x,\nu}$ and if for any $\beta = \nu + p\delta + q\pi$ corresponding to a point in $\Gamma_{x,\nu}$, β_α^- is not in the graph.

A Cocover must be a Corner

Proposition

If $y = s_\alpha x$ is a cocover of x, then $\alpha = \nu + r\delta + j\pi$ must correspond to a corner in the graph $\Gamma_{x,\nu}$.

Proof.

Suppose α is not a corner of $\Gamma_{x,\nu}$. Then there is some $\beta = \nu + p\delta + q\pi$ such that $\beta \neq \alpha$, $\beta \in \Gamma_{x,\nu}$, and $\beta_\alpha^- \in \Gamma_{x,\nu}$. $\beta \in \Gamma_{x,\nu}$ so $\beta > 0$ and $x^{-1}(\beta) < 0$. $\beta_\alpha^- \in \Gamma_{x,\nu}$ so $-s_\alpha(\beta) > 0$ and $-x^{-1}(s_\alpha(\beta)) < 0$. This shows that $\beta \in L_{x,\alpha}$, so y is not a cocover of x. □

What this tells us about cocovers

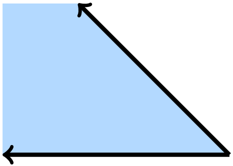

Figure: $\Gamma_{x,\nu}$

- There are finitely many corners and so finitely many cocovers for a given $x \in W$.
- For $\alpha = \nu + r\delta + j\pi$ to be a corner, there are four possibilities for j.

New Theorem

Setup:

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2. ζ is dominant and $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, 2, \ldots, n$.
3. $\tilde{\alpha}$ is an affine root such that $\alpha = -\tilde{v}(\tilde{\alpha}) + j\pi$ is positive

Theorem

With the setup given above, $y = s_\alpha x$ is a cocover of x if and only if one of the following holds:

1. $j = 0$ *and* $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1$.
2. $j = 1$ *and* $\ell(\tilde{v}) = \ell(\tilde{v}s_{\tilde{\alpha}}) + 1 - \langle \tilde{\alpha}, 2\rho \rangle$.
3. $j = \langle \zeta, \tilde{\alpha} \rangle$ *and* $\ell(\tilde{w}^{-1}\tilde{v}s_{\tilde{\alpha}}) = \ell(\tilde{w}^{-1}\tilde{v}) + 1$.
4. $j = \langle \zeta, \tilde{\alpha} \rangle - 1$ *and* $\ell(\tilde{w}^{-1}\tilde{v}s_{\tilde{\alpha}}) = \ell(\tilde{w}^{-1}\tilde{v}) + 1 - \langle \tilde{\alpha}, 2\rho \rangle$.

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