Cohomology of the Cotangent Bundle to a Grassmannian and Puzzles

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Based on work done with Allen Knutson at Cornell University

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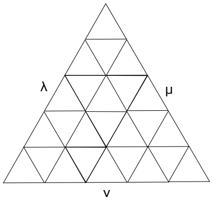
Schubert calculus background

- The set of **Schubert classes** $\{S_\lambda\}$ form a basis over $\mathbb{Z}$ for the cohomology ring $H^*(Gr_k(\mathbb{C}^n))$ where λ is a string with k 1s and $n-k$ 0s.
- Then write
$$S_\lambda S_\mu = \sum_\nu c_{\lambda\mu}^\nu S_\nu$$
- Determining these $c_{\lambda\mu}^\nu$, called **Littlewood-Richardson coefficients** is one of the goals of Schubert calculus
- One way to compute them involves tiling of equilateral triangles called puzzles.

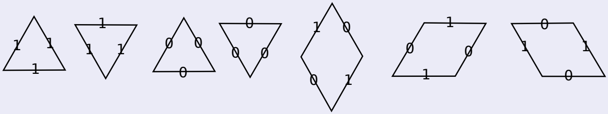
Knutson-Tao puzzles

Suppose you have an equilateral triangle of side length n with the strings of 0s and 1s λ, μ, and ν labeling the NW, NE and S boundaries respectively, all from left to right. This is called a "$\Delta_{\lambda\mu}^{\nu}$ puzzle."



Theorem (KT)

The Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ is the number of ways to tile a $\Delta_{\lambda\mu}^{\nu}$ puzzle with the following puzzle pieces.



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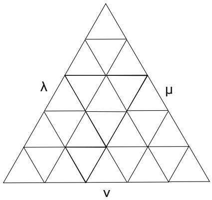
Suppose you have an equilateral triangle of side length n with the strings of 0s and 1s λ, μ, and ν labeling the NW, NE and S boundaries respectively, all from left to right. This is called a "$\Delta^{\nu}_{\lambda\mu}$ puzzle."



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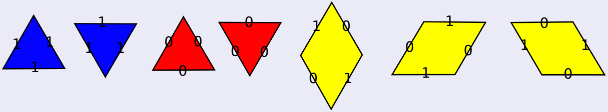
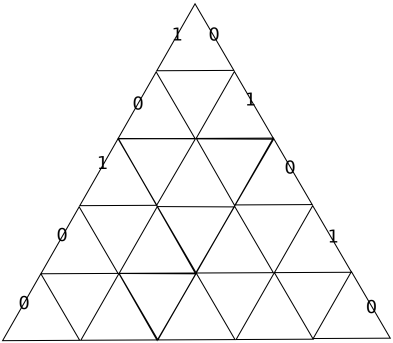


Example



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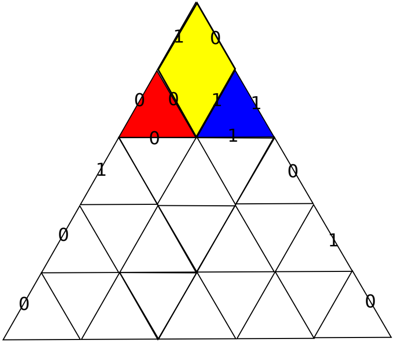
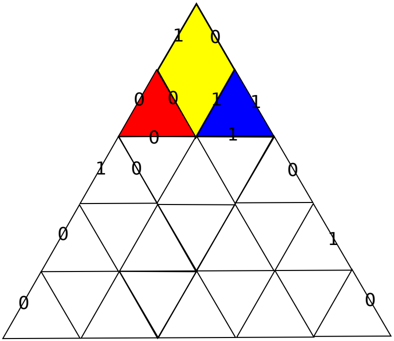
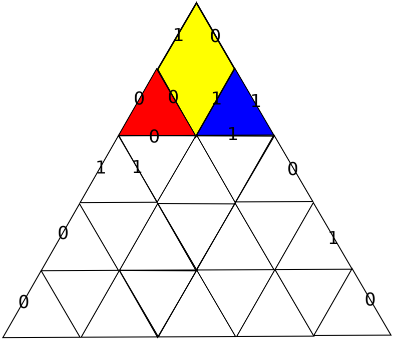


Example





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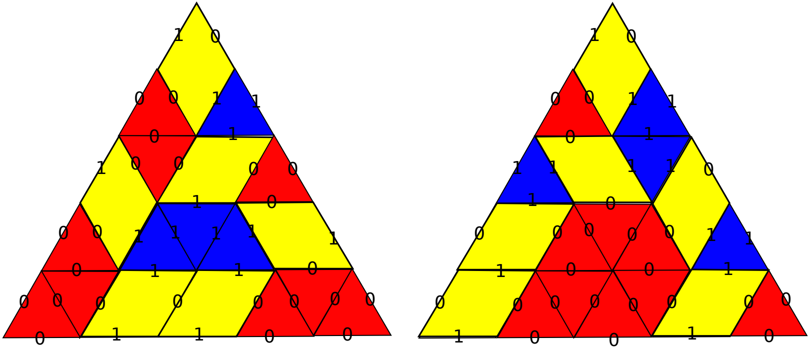
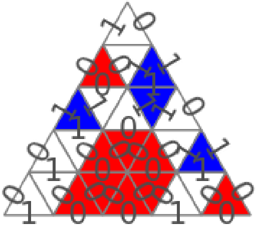
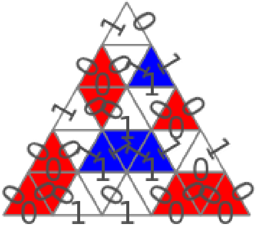


Why use puzzles?

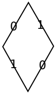
- They're combinatorial. They are a simple visual tool that you can use to attack some very complex problems.
- It provides a positive way to determine L-R coefficients
- They show more symmetries than other positive combinatorial rules.
- Easier to generalize to other Schubert calculus problems than other positive combinatorial rules are.
- Since L-R coefficients show up all over the place, puzzles get a lot of use and are implemented in Sage.



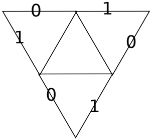


Other puzzle formulas

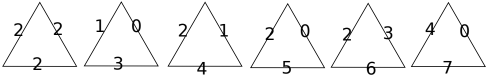
- Equivariant cohomology-Schubert calculus [Knutson, Tao 2001]



- K-theory [Buch '00]



- $H^*(2 - \text{step flag manifolds})$ [Buch, Kresh, Purbhoo, Tamvakis '14]



- $K_T(Gr_k(\mathbb{C}^n))$ [Pechenik, Yong '15]
- Equivariant cohomology of two-step flag varieties [Buch, 15]

Cohomology and Maulik-Okounkov Classes

- In their 2012 paper Quantum Groups and Quantum Cohomology, D. Maulik and A. Okounkov defined the "stable basis" for a class of varieties called Nakajima varieties.
- The Nakajima varieties of a quiver which contains one vertex with no arrows are the cotangent bundles of Grassmann varieties.
- So Maulik and Okounkov's definition describes a basis M_λ for

  $$H^*_{\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n)) \cong H^*(Gr_k(\mathbb{C}^n))[\hbar]$$

  where λ is a string of k 1s and $n-k$ 0s.
- It also describes a basis $\widetilde{M}_\lambda$ in equivariant cohomology:

  $$H^*_{T\times\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n))$$

  These classes form a basis for the space over $\mathbb{Z}[\hbar, y_1, \ldots, y_n]$ after inverting $\hbar$.

Maulik-Okounkov class restrictions

These classes have restrictions $\alpha|_\lambda \in H^*_{T\times\mathbb{C}^\times}$ to fixed points $\mathbb{C}^\lambda \in (T^*Gr_k(\mathbb{C}^n))^{T\times\mathbb{C}^\times}$ of the torus action which satisfy

1. $\widetilde{M}_\lambda|_\mu = 0$ for $\mu \not\geq \lambda$
2. $\widetilde{M}_\lambda|_\lambda = \prod_{i\in[1,k],j\in[1,n-k]} \begin{cases} y_i - y_j & (i,j) \in \lambda \\ \hbar - (y_i - y_j) & (i,j) \notin \lambda \end{cases}$
3. $\hbar \Big| \widetilde{M}_\lambda|_\mu$ for $\mu > \lambda$

These conditions uniquely determine this basis $\{\widetilde{M}_\lambda\}$. Maulik-Okounkov classes are related to each other by a "deformed reflection operator," R_i

$$R_i \cdot \widetilde{M}_\lambda = \widetilde{M}_{r_i\cdot\lambda} \text{ where } R_i = r_i + \hbar\partial_i$$

Product Structure in Equivariant Cohomology

Given that

$$\widetilde{M}_\lambda \cdot \widetilde{M}_\mu = \sum_\nu c^\nu_{\lambda\mu} \widetilde{M}_\nu$$

what are the coefficients $c^\nu_{\lambda\mu} \in \mathbb{Z}[\hbar, y_1, \dots, y_n]$? This is the question my research has been attempting to answer.

Finding the answer in the equivariant case will provide the coefficients in regular cohomology as well, by simply setting $y_i = 0$.

Recall that a Grassmannian, $Gr_k(\mathbb{C}^n)$, is the set of k-planes in $\mathbb{C}^n$. I began by looking at the projective case, i.e. where $k = 1$ or $k = n - 1$. We will use $k = n - 1$.

The case of $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n))$

In $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n))$, the $\widetilde{M}_\lambda$ can be indexed by λ which are strings of $n-1$ 1s and one 0, which we will call $\binom{[n]}{n-1}$. We will use $\widetilde{M}_i$ to mean the class given by the element of $\binom{[n]}{n-1}$ where the 0 is in the ith spot.

Looking at this case our restriction formulas tell us

$$\widetilde{M}_i|_a = \prod_{b\in[1,i)} (y_a - y_b) \prod_{b\in(i,n]} (\hbar + y_a - y_b)$$

We can use a standard inner product on our ring to get a formula for a dual basis:

$$\widetilde{M}_i^*|_a = \prod_{b\in[1,i)} (\hbar + y_a - y_b) \prod_{b\in(i,n]} (y_a - y_b)$$

The case of $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n))$

Theorem

(C) Consider $\lambda, \mu, \nu \in \binom{[n]}{n-1}$ so that the 0 is in the ith, jth, and kth spots respectively. Here c_{ij}^k corresponds to the coefficient for $\widetilde{M}_k$ in $\widetilde{M}_i \widetilde{M}_j$. Then, using equivariant localization, we get

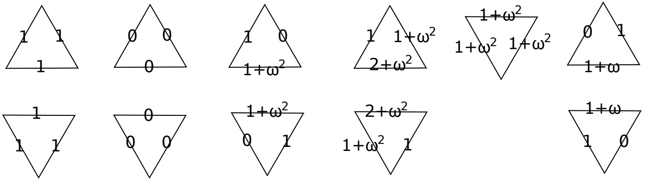
$$c_{ij}^k = \sum_{i,j \leqslant a \leqslant k} \frac{\hbar \prod_{b<i}(y_a - y_b) \prod_{b>i}(\hbar + y_a - y_b) \prod_{b<j}(y_a - y_b) \prod_{b>j}(\hbar + y_a - y_b) \prod_{b>k}(y_a - y_b)}{\prod_{b \neq a}(y_a - y_b) \prod_{b \geqslant k}(\hbar + y_a - y_b)}$$

- NOT positive
- not obviously polynomial

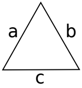
I've been attempting to find a positive combinatorial rule which will be able to compute these product structure coefficients in a more reasonable time frame.

Initial Puzzle Formula for $H^*_{T\times\mathbb{C}^\times}(T^*Gr_{n-1}(\mathbb{C}^n))$

By looking at small examples I was able to come up with the following puzzle pieces



Note that the above puzzle pieces satisfy the boundary label condition that $a + b\omega + c\omega^2 = 0$



However weights are now no longer assigned to pieces. They are assigned to **fiefdoms** which are smallest collections of puzzle pieces with 1s and 0s on the boundary.

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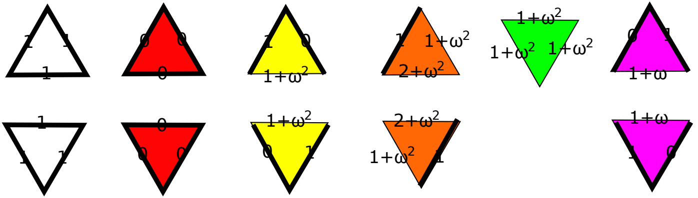

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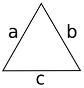

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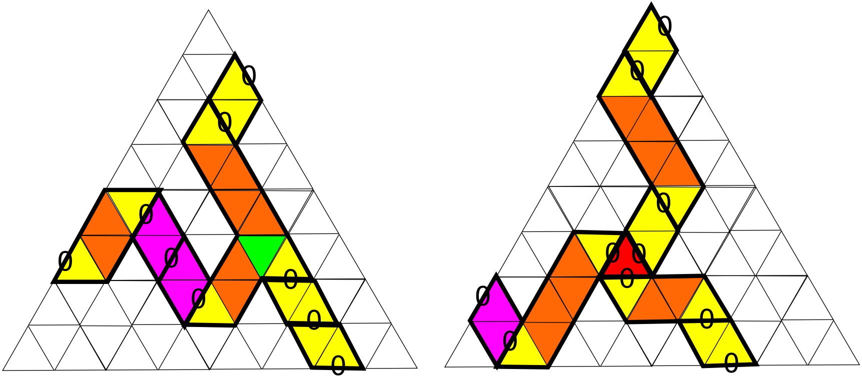
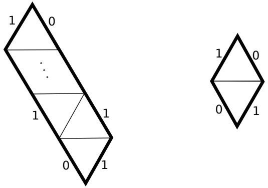
Equivariant puzzles



There is always one central piece with three tendrils coming out that track where the 0 goes within the puzzle.

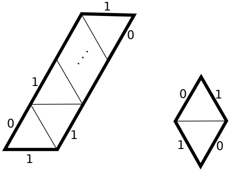
Puzzle pieces and their weights

In the NE tendril, you get fiefdoms with two possible weights:



$\hbar$ $\qquad$ $\hbar - (y_a - y_b)$

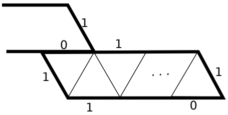
In the NW tendril, you get fiefdoms you also get two possible weights:



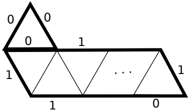
$\hbar$ $\qquad$ $(y_a - y_b)$

Puzzle pieces and their weights

In the S tendril the weight of the fiefdom depends on what kind of fiefdom is above it:

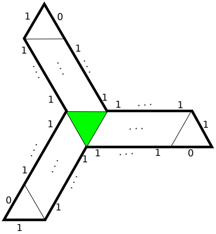


$\hbar + (y_a - y_b)$

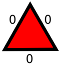


$\hbar$

There are two kinds of central fiefdoms, again with two possible weights:

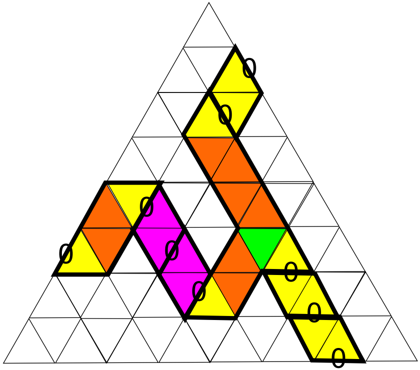


$\hbar$



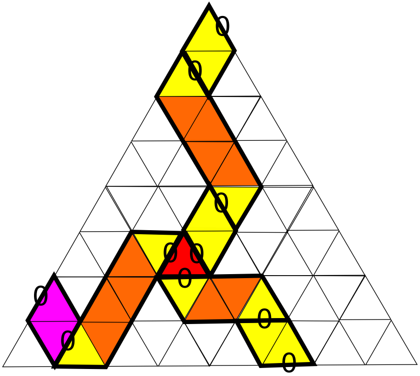
1

Weight of whole puzzle



$$\hbar^2(y_5 - y_2)(y_5 - y_3)(\hbar - (y_8 - y_2))(\hbar + y_7 - y_6)(\hbar + y_7 - y_5)$$

Weight of whole puzzle



$$\hbar^3(y_2 - y_1)(\hbar - (y_8 - y_1))(\hbar - (y_6 - y_3))(\hbar + y_6 - y_5)$$

Puzzle Recurrence Relations

The puzzle weight summations $p_{i,j}^k(\ell, n)$ satisfy the recurrence relations

(1) For $j < n-1$

$$p_{i,j}^n(1,n) = p_{i,j}^n(0,n) - p_{i,j}^{n-1}(0,n)$$

(2) For $\ell > 1$ and $i < n$

$$p_{i,j}^n(\ell,n) = p_{i,j}^n(\ell-1,n) - \prod_{b\in[1,\ell-1]} \frac{\hbar + y_n - y_{n-b}}{\hbar + y_{n-1} - y_{n-b-1}} \cdot A$$

where

$$A = (\hbar + y_1 - y_n)p_{i,j}^{n-1}(\ell-1,n-1) + (\hbar + y_{n-1} - y_{n-\ell})p_{i,n-\ell}^{n-1}(\ell-2,n-1) + \hbar \cdot \sum_{a\in[2,n-\ell-1]} p_{i,a}^{n-1}(\ell-1,n-1)$$

where $p_{i,j}^k(\ell, n)$ is the sum of the weights corresponding to n-dimensional puzzles with 0s on the boundary at i, j and k with **at least** ℓ copies of the $(1,0,1,0)$ sideways rhombus stacked at the bottom of the southern tendril.

This has been checked by computer for up to $n = 9$.

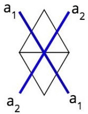
A Different Proof Method using R-matrices

In 2017, Knutson and Zinn-Justin found new proofs of the already existing puzzle formulas for $H^*(Gr_k(\mathbb{C}^n))$, 2-step flag manifolds, and 3-step manifolds, as well as for two previously unsolved Schubert calculus problems : K(2-step flag manifolds) and K(3-step flag manifolds).

Let V be a finite-dimensional vector space, and $a, b, c, \in \mathbb{C}$ parameters. Then the algebraic formulation of the **(rational) Yang-Baxter equation** on $R \in End(V \otimes V)(u)$ is

$$R_{12}(a-b)R_{13}(a-c)R_{23}(b-c) = R_{23}(b-c)R_{13}(a-c)R_{12}(a-b)$$

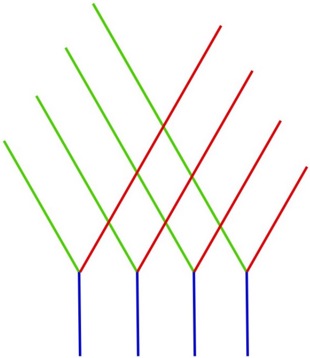
Jimbo and Drinfeld constructed solutions of the YBE in the quantized loop algebra $U_q(\mathfrak{g}[z^{\pm}])$. These solutions are called R-**matrices**, and they provide an isomorphism from the tensor product $(V, a_1) \otimes (V, a_2)$ to $(V, a_2) \otimes (V, a_1)$.



How does this help?

So for $\lambda, \mu, \nu \in \binom{[n]}{k}$ representing an element of our basis, all of the crossings in the following diagram are encoded by the corresponding R-matrix

$$\widetilde{M}_\lambda \cdot \widetilde{M}_\mu = \sum_\nu c_{\lambda\mu}^\nu \widetilde{M}_\nu$$



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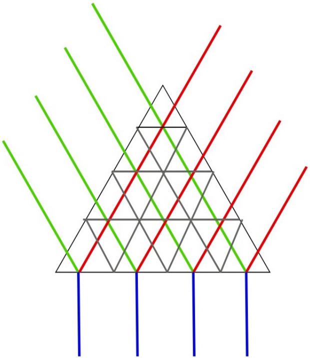
$$\widetilde{M}_\lambda \cdot \widetilde{M}_\mu = \sum_\nu c^\nu_{\lambda\mu} \widetilde{M}_\nu$$

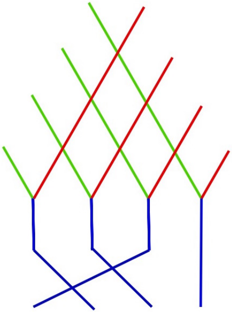


This picture is the dual of a puzzle!

How does this help?

If we allow σ shaped partitions at the bottom of this picture, then it will encode the entire right side of the equivariant localization formula for our product.

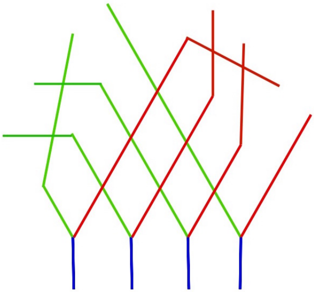
$$\widetilde{M}_\lambda\Big|_\sigma \cdot \widetilde{M}_\mu\Big|_\sigma = \sum_\nu c^\nu_{\lambda\mu}\, \widetilde{M}_\nu\Big|_\sigma$$



Rearranging our picture

We can move our partition σ through the puzzle by showing that the puzzles satisfy certain equations (one in particular being a visual version of the YBE). This leads us to the following picture which encodes the left hand side of our product formula

$$\widetilde{M}_\lambda\Big|_\sigma \cdot \widetilde{M}_\mu\Big|_\sigma = \sum_\nu c^\nu_{\lambda\mu}\ \widetilde{M}_\nu\Big|_\sigma$$



Results from KZJ and ideas for $H^*_{\mathbb{C}^\times}(T^*Gr_k(\mathbb{C}^n))$

So this new proof method involves showing that the puzzles and dual puzzles satisfy several equations, as well as finding the R-matrix which encodes the correct products. Knutson and Zinn-Justin have done this in the following cases:

$$\begin{array}{ccc} H^*(Gr_k(\mathbb{C}^n)) & \longleftrightarrow & U_q(\mathfrak{sl}_3[z^\pm]) \circlearrowright \mathbb{C}^3 \\ 2\text{ - step} & \longleftrightarrow & U_q(\mathfrak{so}_8[z^\pm]) \circlearrowright \mathbb{C}^8 \\ 3\text{ - step} & \longleftrightarrow & U_q(\mathfrak{e}_6[z^\pm]) \circlearrowright \mathbb{C}^{27} \end{array}$$

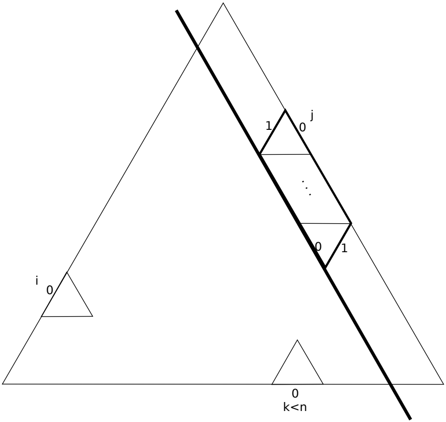
My initial computations have indicated the formula will include at least seven edge labels:

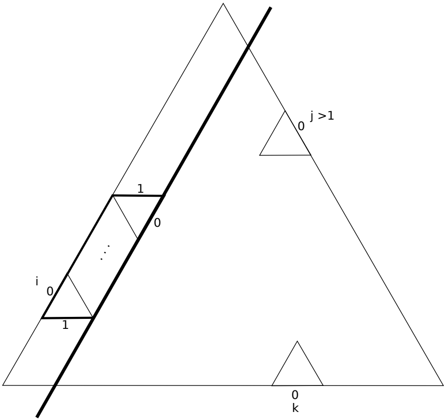
$$0, 1, \omega^2, 1+\omega^2, 2+\omega^2, 1+2\omega^2, 2+2\omega^2$$

This indicates that the correct R-matrix may be $U_q(\mathfrak{g}_2[z^\pm]) \circlearrowright \mathbb{C}^7$, which is what I am investigating now.

The End

Inductively reducing to $c_{i,1}^{n}$





Definition for proof

Define $p_{i,j}^k(\ell, n)$ as the sum of the weights corresponding to n-dimensional puzzles with 0s on the boundary at i, j and k with **at least** ℓ copies of the $(1,0,1,0)$ sideways rhombus stacked at the bottom of the southern tendril.

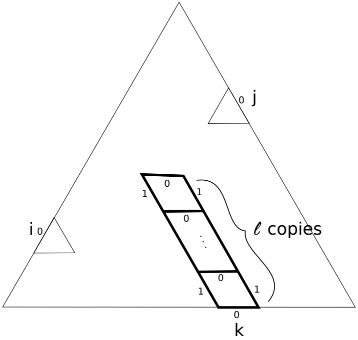


Illustration of $p_{i,j}^k(\ell, n)$

Main Conjecture

Using the above definition $c_{i,j}^k = p_{i,j}^k(0, n)$*, i.e. the total weight of* ***all*** *puzzles with the right boundary.*

Relating $c^n_{i,1} = p^n_{i,1}(0, n)$ to smaller puzzles

Lemma

The puzzle weight summations $p^k_{i,j}(\ell, n)$ satisfy the recurrence relations

(1) *For $j < n-1$*

$$p^n_{i,j}(1, n) = p^n_{i,j}(0, n) - p^{n-1}_{i,j}(0, n)$$

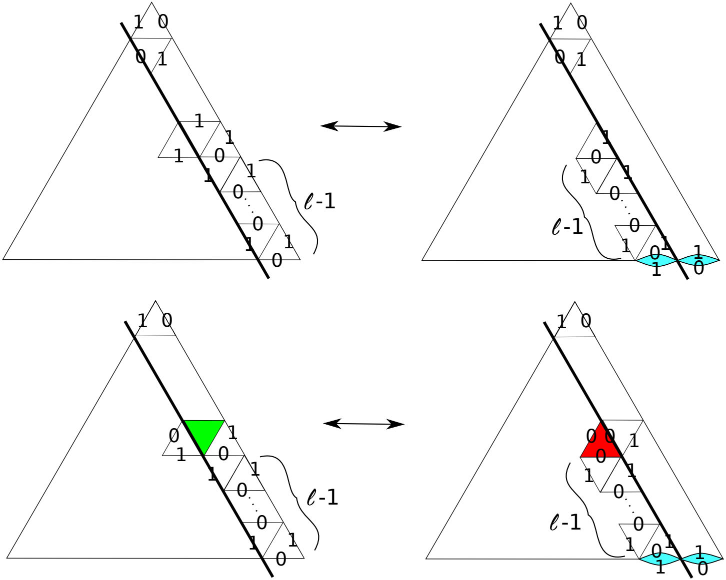
(2) *For $\ell > 1$ and $i < n$*

$$p^n_{i,j}(\ell, n) = p^n_{i,j}(\ell - 1, n) - \prod_{b \in [1, \ell-1]} \frac{\hbar + y_n - y_{n-b}}{\hbar + y_{n-1} - y_{n-b-1}} \cdot A$$

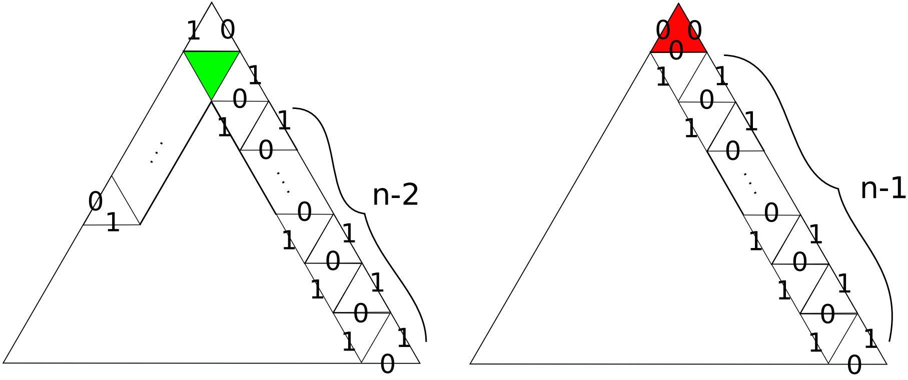
where

$$\begin{aligned} A &= (\hbar + y_1 - y_n) p^{n-1}_{i,j}(\ell - 1, n-1) + (\hbar + y_{n-1} - y_{n-\ell}) p^{n-1}_{i,n-\ell}(\ell - 2, n-1) \\ &\quad + \hbar \cdot \sum_{a \in [2, n-\ell-1]} p^{n-1}_{i,a}(\ell - 1, n-1) \end{aligned}$$

Gash argument



$p^{n}_{i,1}(n-2,n)$



$$\hbar \cdot \prod_{b\in[1,n-2]} (\hbar + y_n - y_{n-b})$$

Rational function

Definition

Let

$$r_{i,j}^{n}(0,n) := c_{i,j}^{n}$$

as given by the rational function formula. Then we can define $r_{i,j}^{n}(\ell,n)$ for any $\ell < n-j$ by using the p recurrence relations from 3 slides ago.

The main conjecture reduces to this:
Use the recurrence relations and the exact formula for $c_{i,j}^{k}$ to inductively define the a priori rational function $r_{i,1}^{n}(n-2,n)$ and show that

$$r_{i,1}^{n}(n-2,n) = \hbar \cdot \prod_{b\in[1,n-2]} (\hbar + y_n - y_{n-b}) \qquad \big(\text{that being } p_{i,1}^{n}(n-2,n)\ \big)$$

This has been checked by computer for up to $n = 9$.