Topology Review Guide, June 2005.

Note: The Top. general syllabus* was significantly revised in 2004. Thus some problems from before 2004 have been modified or deleted.

Topology General Exam
Fall 2004

Guidelines This is a 4 hour exam, and "closed book".

1. Let X be a topological space with closed points. Recall that X is *normal* if whenever A and B are disjoint closed subsets of X, there exist two disjoint open sets containing A and B respectively.

(a) Let X be normal. Given $C \subset U \subset X$ with C closed and U open, show that there exists an open set W such that $C \subset W$ and $\overline{W} \subset U$.

(b) Let $\{U_1, \ldots, U_n\}$ be a finite open cover of a normal space X. Show that there exists another open cover $\{V_1, \ldots, V_n\}$ such that $\overline{V}_i \subset U_i$ for all i.

2. **(a)** Let X be compact and Y Hausdorff. If $g : X \to Y$ is continuous, show that its image $g(X)$, viewed as *subspace* of Y, has the same topology as $g(X)$ viewed as a *quotient space* of X: check that a subset of $g(X)$ is closed in $g(X)_{sub}$ iff it is closed in $g(X)_{quo}$.

(b) Suppose now that Y is not just Hausdorff, but is normal, and suppose that $h : X \to \mathbb{R}$ is a continuous function with the property that $g(x_1) = g(x_2)$ implies $h(x_1) = h(x_2)$. Show that then h factors through g: there exists a continuous $f : Y \to \mathbb{R}$ such that $h = f \circ g$.

3. Let G be a topological group.

(a) If H is an *open* subgroup of G, show that it is also a *closed* subgroup.

(b) If G is connected, and H is a subgroup that is both discrete (as a subspace) and normal (in the group theory sense), show that H is a *central* subgroup of G: $gh = hg$ for all $h \in H, g \in G$.

4. **(a)** Compute the fundamental group of $S^2 \vee \mathbb{R}P^2$ (the wedge sum of S^2 and $\mathbb{R}P^2$).

(b) Describe the universal cover of $S^2 \vee \mathbb{R}P^2$.

(c) Compute the homology groups of both $S^2 \vee \mathbb{R}P^2$ and its universal cover.

* The topology general exam syllabus is available on the Math Dept. grad program website.

5. Let X be a topological space, and $e \in X$. An *H–space structure* on (X, e) is a continuous map $m : X \times X \to X$ such that $m(x, e) = x = m(e, x)$ for all $x \in X$.

Suppose X is path connected and locally path connected, with universal cover $p : \widetilde{X} \to X$. Suppose $m : X \times X \to X$ is an H–space structure on (X, e), and $\tilde{e} \in p^{-1}(e)$. Show that there exists a unique H-space structure on $(\widetilde{X}, \tilde{e})$, $\widetilde{m} : \widetilde{X} \times \widetilde{X} \to \widetilde{X}$, such that the diagram

$$\begin{array}{ccc} \widetilde{X} \times \widetilde{X} & \xrightarrow{\widetilde{m}} & \widetilde{X} \\ \downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p} \\ X \times X & \xrightarrow{m} & X \end{array}$$

commutes.

6. A space X has the *fixed point property* if every continuous self map $f : X \to X$ has at least one fixed point.

Do the following spaces have the fixed point property? (Justify your answer, of course.)

(a) S^1 (b) D^2 (c) S^2 (d) $\mathbb{R}P^2$ (e) $\mathbb{R}P^3$.

7. Let Y be a space with rank $H_4(Y)$ = rank $H_5(Y) = 3$. Suppose Z is obtained from Y by attaching two 5–dimensional cells. What are the values that the pair (rank $H_4(Z)$, rank $H_5(Z)$) might take?

8. Let $f_*, g_* : C_* \to D_*$ be two chain maps between two chain complexes.

(a) Define: f_* is *chain homotopic* to g_*.

(b) Prove: if f_* is chain homotopic to g_*, then $H_*(f_*) = H_*(g_*) : H_*(C_*) \to H_*(D_*)$.

Fall 2003

Topology general exam

Solve 7 of the following 8 problems.

1. (a) Describe the fundamental group and universal cover of the torus $S^1 \times S^1$.

(b) Prove that any [redacted] continuous map from the two-sphere S^2 to the torus ~~has (mod 2) degree equal to zero~~. is 0 on H_2.

2. Let M be a 2-dimensional submanifold of $\mathbb{R}^3$, and let $d \colon M \longrightarrow \mathbb{R}$ be the distance to the origin. Suppose the origin lies in the complement of M. Show that the critical points of d are precisely the points of M where M is tangent to some sphere centered at the origin.

3. Let $(X, \mathcal{T})$ be a topological space. Let Y be the union of X and a point p (not in X). Let $\mathcal{S}$ be the collection of subsets of Y given by

(1) if $U \subset Y$ and $p \notin U$, then $U \in \mathcal{S}$ if and only if U is open in X,

(2) if $U \subset Y$ and $p \in U$, then $U \in \mathcal{S}$ if and only if $Y \setminus U$ is closed and compact in X.

Show that:

(1) $\mathcal{S}$ is a topology for Y.
(2) Y is compact.
(3) X is an open subset of Y and the topology induced on X from $\mathcal{S}$ is $\mathcal{T}$.
(4) X is dense in Y if and only if $(X, \mathcal{T})$ is not compact.
(5) Y is a T_1 space if and only if X is T_1.
(6) Y is Hausdorff if and only if X is Hausdorff and locally compact.
(7) If $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, \sum_1^n x_i^2 < 1\}$ with the standard topology, prove that Y is homeomorphic to S^n.

4. (a) Let X be a topological space. Let U_1, U_2 be dense open subsets. Prove that their intersection is a dense subset of X.

(b) Let X be a compact Hausdorff space. Let A be a subset of X and let U be an open subset of X such that closure$(A) \subset U$. Prove that there exists an open set W such that closure$(A) \subset W$ and closure$(W) \subset U$.

(c) Let X be a compact Hausdorff space. Let $\{U_n\}$ be a countable collection of dense open subsets of X. Prove that their intersection is a dense subset of X.

(d) Give an example of a compact space X, and a countable collection of dense open subsets of X, such that the intersection is not dense in X.

5. (a) Describe the universal cover of the figure eight $S^1 \vee S^1$.

(b) Describe a non-trivial two-fold covering space of $S^1 \vee S^1$. How many different two-fold covering spaces of $S^1 \vee S^1$ are there?

6. Show that the special linear group $SL(n, \mathbb{R})$ is a smooth manifold.

7. Let M be a non-empty smooth n-dimensional manifold, and $f\colon M \longrightarrow \mathbb{R}$ be a smooth map.

(a) Show that if M is compact then there are elements r in $\mathbb{R}$ which are not regular values.

(b) If $S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_0^n x_i^2 = 1\}$ and $f\colon S^n \longrightarrow \mathbb{R}$ is given by $f(x_0, \ldots, x_n) = x_0^2$, what are the regular values of f? Show that $f^{-1}(r)$ is a submanifold of S^n for all values of r.

8. Prove that any smooth map $f\colon D^n \longrightarrow D^n$ has a fixed point, for any $n \geq 1$.

Topology General Exam **Fall 2002**

Guidelines This is three hours, and "closed book".

1. Let $S_1, S_2, \ldots$ be a sequence of finite sets each having at least two elements. Each S_n is a topological space with the discrete topology. Let

$$X = \prod_{n=0}^{\infty} S_n$$

be given the product topology.

(a) Is X discrete? Hausdorff? Compact? Connected? Normal? Metrizable?

(b) Describe the path connected components of X.

2. Recall that a topological group G is a group that is also a topological space, such that the functions

$$m : G \times G \longrightarrow G \text{ and } i : G \longrightarrow G$$

defined by $m(a, b) = ab$ and $i(a) = a^{-1}$ are continuous.

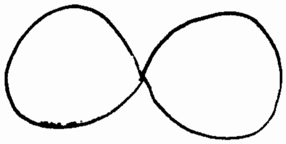
(a) Prove the useful lemma: if G is a topological group, and U is an open neighborhood of the unit e, then e has another open neighborhood W such that

$$a, b \in W \Rightarrow a^{-1}b \in U.$$

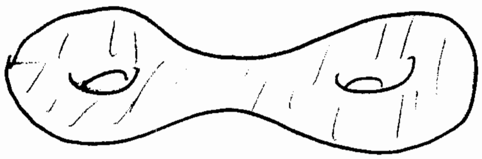
(b) Suppose the topological group G is connected, and $H \subset G$ is a *discrete* subgroup, i.e. a subgroup which is discrete with the subspace topology. Let $p : G \to G/H$ be the projection on the space of cosets, i.e. $p(g) = gH$, and give G/H the quotient topology. Show that p is a covering space map.

(c) With G and H as in (b), how are the fundamental groups $\pi_1(G, e)$ and $\pi_1(G/H, eH)$ related?

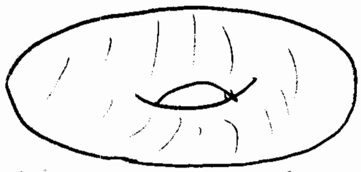
3. As a space covered by $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$ has the structure of a ~~smooth manifold~~ quotient space. The circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is a smooth ~~submanifold~~ subset of $\mathbb{R}^2$. Prove the intuitively obvious fact: $\mathbb{R}/\mathbb{Z}$ is ~~diffeomorphic~~ homeomorphic to S^1.

4. Let R be the figure eight space: 

(a) Explain (e.g. with convincing pictures) why R is a retract of the genus 2 surface:



(b) In contrast, prove that R is *not* a retract of the torus $S^1 \times S^1$:



5. Recall that $\mathbb{R}P^2 = S^2/(\sim)$, where $(x, y, z) \sim (-x, -y, -z)$ defines the equivalence relation. Write down an explicit smooth atlas for $\mathbb{R}P^2$, exhibiting it as a 2 dimensional smooth manifold. Remark: your atlas will need at least three charts.

6. Let M be a smooth manifold of dimension n, and $f : M \to \mathbb{R}^N$ a smooth map with $N > 2n$.

(a) Let $g : TM \to \mathbb{R}^N$ be defined by $g(v) = df_x(v)$ for $v \in T_xM$. Explain why g cannot be onto.

(b) Let $v \in \mathbb{R}^N$ be chosen to *not* be in the image of g, and let $L : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be a surjective linear map satisfying $L(v) = 0$. Show that, if the original map f is an immersion, then so is $L \circ f : M \to \mathbb{R}^{N-1}$.

7. **(a)** Give the definition of a 1–form on a smooth manifold M.

(b) Show that the vector space of all 1–forms on S^1 is isomorphic to the vector space of all functions $f : S^1 \to \mathbb{R}$.

[redacted]

General Examination **Topology** **Fall 2001**

Guidelines: 3 hours. Do 4 (or more) of the 7 problems.

1. The topological space Z is the union of two closed subspaces X and Y.

   (a) Show that $C \subset Z$ is closed if and only if $X \cap C$ is closed in X and $Y \cap C$ is closed in Y.

   (b) Show that if X and Y are normal, then Z is normal.

2. Let G be a topological group and $H \subset G$ a subgroup. Let $\pi : G \to G/H$ be the quotient map with G/H having the quotient topology.

   (a) Show that π is an open map.

   (b) If H is compact, show that π is a closed map.

3. (a) Show that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(x, x_0) \times \pi_1(Y, y_0)$.

   (b) If G is a topological group, with unit $e \in G$, show that $\pi_1(G, e)$ is abelian.

   (c) Exhibit a space X for which $\pi_1(X, x_0)$ is not abelian.

4. Let X and Y be nice spaces (connected, locally path connected, semilocally simply connected) with Y a subspace of X and $i : Y \to X$ the inclusion. Let $\pi : \tilde{X} \to X$ be a covering space with $\tilde{e} \in \tilde{X}$, $\pi(\tilde{e}) = e \in Y$. Let $\tilde{Y} \subset \tilde{X}$ be the path component of $\pi^{-1}(Y)$ containing $\tilde{e}$. What is $\pi_1(\tilde{Y}, \tilde{e})$ in terms of $\pi_1(Y, e)$, $\pi_1(X, e)$, $\pi_1(\tilde{X}, \tilde{e})$?

Topology General Exam **Fall 2000**

Guidelines This is four hours, and "closed book".

1. A topological space X is said to be *Noetherian* if every open set in X is compact. (The name is derived from the fact that the open sets of such spaces satisfy the ascending chain condition.)

(a) Show that every subspace of a Noetherian topological space is again Noetherian.

(b) Show that, if X is Noetherian, and $f : X \to Y$ is continuous and onto, then Y is Noetherian.

(c) Show that if X is Noetherian and Hausdorff, then X is a discrete space with a finite number of points.

(d) Let X be a set with the *finite complement topology*: the open sets in X are the empty set and complements of finite subsets. Check that X with this topology is Noetherian.

(e) Show by example that an infinite product of Noetherian topological spaces need not be Noetherian.

2. Recall that a group G acts on a topological space X means that there are continuous maps

$$g\cdot : X \to X,$$

for all $g \in G$, such that $g \cdot (h \cdot x) = gh \cdot x$ and $e \cdot x = x$ for all $g, h \in G$, $x \in X$. (e is the identity element in G.) The action is *free* if $g \cdot x = x$ for any $x \in X$ implies that $g = e$. We let X/G denote the quotient space of orbits, i.e. $X/(\sim)$, where $x \sim y$ if there exists $g \in G$ with $g \cdot x = y$.

(a) Show that if a finite group G acts freely on a Hausdorff space X, then the quotient map $p : X \to X/G$ is a covering map (i.e. X is a covering space of X/G via p).

(b) In the situation of part (a), suppose that X is also path connected and simply connected. Show that the fundamental group of X/G is isomorphic to G.

3. Let X be a compact topological space and Y be normal. Suppose $g : X \to Y$ and $h : X \to \mathbf{R}$ are continuous functions such that, for all $x_1, x_2 \in X$, if $g(x_1) = g(x_2)$, then $h(x_1) = h(x_2)$. Show that then h factors through g: there exists a continuous function $f : Y \to \mathbf{R}$ such that $h = f \circ g$.

4. Let $F : \mathbf{R}^3 \to \mathbf{R}^2$ be $F(x, y, z) = (x^2 + y^2, z^2 + 1)$, and let $f : S^2 \to \mathbf{R}^2$ be the restriction of F to the sphere S^2.

(a) Show that F is transversal to the diagonal $\Delta(\mathbf{R}) \subset \mathbf{R}^2$.

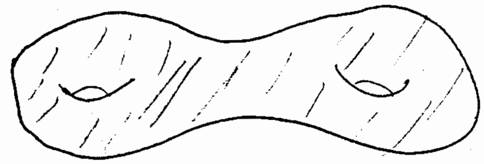
(b) Is f still transversal to $\Delta(\mathbf{R})$?

5. Let M_1, M_2, and N be closed smooth manifolds all of dimension n. (closed = compact, with empty boundary.) We say that two smooth maps $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are *cobordant* if there exists a smooth compact $n+1$–dimensional manifold W with $\partial W = M_1 \coprod M_2$, and a smooth function $F : W \to N$ such that $F|_{M_i} = f_i$.

Show that cobordant maps have the same mod 2 degree.

6. **(a)** Let T be the torus $S^1 \times S^1$. Describe two vector fields v and w on T such that $v(x) \neq w(x)$ for all $x \in T$.

(b) Let M be the closed surface of genus two:



Let v and w be smooth vector fields on M. Explain why there must exist at least one point $x \in M$ with $v(x) = w(x)$.

Extra
Point Set Exercises

1. Projections from products
Prove *one* of the following statements, and give a *counterexample* to the other.
(a) The projection $\pi_1 : X \times Y \longrightarrow X$ is an "open" map, i.e. $\pi_1(\text{open in } X \times Y)$ is always open in X.
(b) The projection $\pi_1 : X \times Y \longrightarrow X$ is a "closed" map, i.e. $\pi_1(\text{closed in } X \times Y)$ is always closed in X.

2. The diagonal and the Hausdorff property
Let $\Delta(X) = \{(x,x) \mid x \in X\} \subset X \times X$. Show that $\Delta(X)$ is a closed subset of $X \times X \Leftrightarrow X$ is Hausdorff (T_2).

3. Fixed point sets
If X is Hausdorff and $f : X \longrightarrow X$ is continuous, show that the set of "fixed points", $F = \{x \in X \mid f(x) = x\}$ is a closed set in X. (Hint: use problem 2.)

4. Dense sets and continuous functions
Let A be a dense subset of X, and let Y be Hausdorff. Show that a continuous function from X to Y is determined by its restriction to A, i.e. show that if $f, g : X \longrightarrow Y$ are continuous and $f|_A = g|_A$ then $f = g$. (Said differently, this problem says that if a continuous function from A to Y extends to X, then it extends *uniquely*.)

5. Exercise 9, section 2-7 of Munkres, but please assume "Y" is **R**.

6. Topological Groups
A *topological group* G is a group that is also a topological space, such that the functions

$$m : G \times G \longrightarrow G \text{ and } i : G \longrightarrow G$$

defined by $m(x,y) = xy$ and $i(x) = x^{-1}$ are continuous.
Notation: If A and B are subsets of G, $A \cdot B = \{xy \mid x \in A, y \in B\}$ and $A^{-1} = \{x^{-1} \mid x \in A\}$. $e \in G$ is the identity element of the group.

(a) Show that if $U \subset G$ is open, so is $A \cdot U$ (and $U \cdot A$), for any subset $A \subset G$. (Hint: you may wish to first do problem 3 on page 144 of Munkres.)
(b) Show that if H is a subgroup of G, the quotient map $q : G \longrightarrow G/H$ is open.
(c) Show that if H is a subgroup of G so is its closure $\overline{H}$. In other words, check that if $x, y \in \overline{H}$, then $xy \in \overline{H}$ and $x^{-1} \in \overline{H}$.
(d) Show topological groups satisfy the "regularity" axiom: this is problem 6 on page 145 of Munkres, and I strongly recommend his three part approach.

7. Connected sets. Exercise 3, §3-1 of Munkres.

8. Homotopic maps.

Let X and Y be topological spaces. We say two continuous functions $f, g : X \longrightarrow Y$ are *homotopic* if there exists a continuous function $H : X \times [0,1] \longrightarrow Y$ such that $H(x,0) = f(x)\ \forall x \in X$ and $H(x,1) = g(x)\ \forall x \in X$. In this case, we write $f \simeq g$. (The intuitive meaning here is that f can be continuously deformed into g, as H defines a whole family of continuous functions $H_t : X \longrightarrow Y$, for $t \in [0,1]$, by the formula $H_t(x) = H(x,t)$, with $H_0 = f$ and $H_1 = g$.)

(a) Show that $\simeq$ is an equivalence relation on the set of continuous functions from X to Y.

(b) If X is a single point, explain why the homotopy equivalence classes of maps from X to Y can be thought of as the path components of Y.

Remark A famous problem is to determine the homotopy equivalence classes of maps from one sphere to another. For example, there turn out to be exactly 240 equivalence classes of maps from S^{16} to S^9. This problem has been much studied (including, on occasion, by your instructor), but is only partially understood.

9. Compactness and the Hausdorff property. Exercise 5, §3.5 of Munkres.

This problem has you generalizing the fact that a compact subset of a Hausdorff space is closed.

10. Compactness and closed maps. Exercise 8, §3.5 of Munkres.

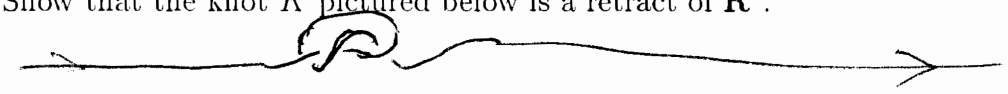
This problem has you show that, if Y is compact, then the projection $X \times Y \longrightarrow X$ will be a closed map.

11. The Tietze Extension Theorem is knot so bad.

Let A be a subspace of a topological space X, with inclusion map $i : A \subset X$. We say that A is a *retract* of X if there exists a continuous map $r : X \to A$ such that $r \circ i = 1_A : A \to A$. (Such a map r is called a *retraction.*)

(a) Show that the x-axis in $\mathbf{R}^3$ is a retract of $\mathbf{R}^3$, by writing down an explicit retraction.

(b) Show that the knot K pictured below is a retract of $\mathbf{R}^3$.



12. Homotopy equivalent spaces.

We say two topological spaces X and Y are *homotopy equivalent* if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that both $f \circ g \simeq 1_Y : Y \to Y$ and $g \circ f \simeq 1_X : X \to X$. (Recall the notation: given two functions $a, b : A \to B$, we write $a \simeq b$ if a and b are homotopic.)

(a) Show that if X is a convex set in $\mathbf{R}^n$ then X is homotopy equivalent to a point. (Jargon: X is *contractible*.)
(b) Show that $\mathbf{R}^2$ – {two points} is homotopy equivalent to the figure eight: (Informal pictures suffice.)
(c) Show that the torus – {point} is also homotopy equivalent to the figure eight. (Again, informal pictures suffice!)

13. Null homotopic maps from spheres.

Notation: Given $A \subset X$, X/A denotes the quotient space $X/(\sim)$, where $a \sim a'$ for all $a, a' \in A$. (Informally, X/A is obtained form X by collapsing A to a point.)
(a) Show that the map $h : S^{n-1} \times [0,1] \longrightarrow B^n$, given by $f(\vec{x}, t) = t\vec{x}$, induces a homeomorphism $\bar{h} : (S^{n-1} \times [0,1])/(S^{n-1} \times \{0\}) \cong B^n$. (Hint: I would strongly urge you to use Theorem 5.6 of Munkres.)
(b) We say that $f : X \longrightarrow Y$ is *null homotopic* if it is homotopic to a constant map. Show that $f : S^{n-1} \longrightarrow Y$ is null homotopic if and only if f extends to a continuous function $\bar{f} : B^n \longrightarrow Y$. (Hint: Use part (a).)

14. Topological groups have abelian fundamental groups.

Show that, if G is a topological group with identity element x_0, then $\pi_1(G, x_0)$ is abelian (i.e. commutative). This is #6 of §8.2 of Munkres, where things are broken down into reasonable (and interesting) steps: (a), (b), (c), (d). But I suggest inserting a step (b$\frac{1}{2}$): check that $(f * f') \otimes (g * g') = (f \otimes g) * (f' \otimes g')$.

15. The infinite union topology.

Let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of spaces, with X_n a subspace of X_{n+1} for each n. Let $X = \bigcup_{n=1}^{\infty} X_n$. Define a topology on X by defining $O \subset X$ to be open in X if each $O \cap X_n$ is open in X_n for each n.

Important examples include $\mathbf{R}^\infty = \bigcup_{n=1}^{\infty} \mathbf{R}^n$, $SO = \bigcup_{n=1}^{\infty} SO(n)$, $S^\infty = \bigcup_{n=1}^{\infty} S^n$, and $\mathbf{RP}^\infty = \bigcup_{n=1}^{\infty} \mathbf{RP}^n$.

(a) Check the basics:

(1) This really *is* a topology.
(2) $C \subset X$ is closed in X if each $C \cap X_n$ is closed in X_n for each n.
(3) X_n is a subspace of X.
(4) A function $f : X \to Y$ is continuous if its restriction to each X_n is continuous.

(b) Assume X_n is T_1 for all n. Suppose one is given points $x_i \in X_{n_i} - X_{n_{i-1}}$, with $n_1 < n_2 < \ldots$. Let S be the set of all these points. Show that S is a discrete subset of X.

(c) Assume X_n is T_1 for all n. Use (b) to prove that if $g : C \longrightarrow X$ is a continuous map from a compact space C, then $g(C) \subset X_n$ for some n.
(d) (A typical application of (c).) Starting from the observation that $S^n \hookrightarrow S^{n+1}$ is null homotopic, conclude that any continuous map from a compact space to S^∞ is null homotopic.