Topology General Exam
August 23, 2013, 9 am - 1 pm.

Name: ______________________________

Instructions: This is a four hour exam and closed book. There are nine problems. To get credit for a problem, you must carefully justify all (nontrivial) claims and show all calculations. You may use without proof anything that is proved in the course textbooks, or other standard reference. If you do so, please refer to the theorem by name or give its statement. You may not cite a statement you are explicitly asked to prove, or facts that were given as exercises or homework.

1. a) Suppose a group G acts on a space X. Give a precise definition of the statement that G acts *properly discontinuously.*

b) Recall that G acts *freely* if for every $x \in X$, the only $g \in G$ with the property that $g \cdot x = x$ is the identity $g = e$. Prove that a finite group G acting freely on a Hausdorff space X acts properly discontinuously.

2. Let M be a compact orientable n-manifold with boundary. It is a fact that for each k, $0 \leq k \leq n$, the groups (vector spaces) $H_k(M;\mathbb{R})$ and $H_{n-k}(M,\partial M;\mathbb{R})$ are isomorphic. Moreover, an isomorphism may be chosen between them such that in the diagram

$$\begin{array}{ccc} H_k(M;\mathbb{R}) & \xrightarrow{\ i\ } & H_k(M,\partial M;\mathbb{R}) \\ \downarrow \sim & & \sim \downarrow \\ H_{n-k}(M,\partial M;\mathbb{R}) & \xleftarrow{\ j\ } & H_{n-k}(M;\mathbb{R}) \end{array}$$

the inclusion-induced maps i and j have matrix representatives that are transposes of each other.

Now suppose M is odd-dimensional, $\dim(M) = 2m+1$. Prove that $\dim H_m(\partial M;\mathbb{R})$ is even, and that the map $H_m(\partial M;\mathbb{R}) \to H_m(M;\mathbb{R})$ has rank equal to $\frac{1}{2}\dim H_m(\partial M;\mathbb{R})$.

3. Let $f : X \to X$ be a continuous map. The *mapping torus* of f is the quotient space

$$T_f = X \times [0,1]/\{(x,1) \sim (f(x),0) \text{ for all } x \in X\}.$$

Find the homology groups $H_i(T_f;\mathbb{Z})$, $i \geq 0$, if $f : S^1 \to S^1$ is a map of degree $n \in \mathbb{Z}$.

4. Suppose the projective plane $\mathbb{R}P^2$ is written as a union $\mathbb{R}P^2 = U_1 \cup \cdots \cup U_n$ where each open set U_i is homeomorphic to $\mathbb{R}^2$. Let $V_i = U_1 \cup \cdots \cup U_i$, for $1 \leq i \leq n$. Prove that there exists $i \leq n$ such that the intersection $U_i \cap V_{i-1}$ is either disconnected or empty.

5. Let X and Y be connected, locally path connected, and semi-locally simply connected, and let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of X and Y, respectively. Prove that if X and Y are homotopy equivalent, then so are $\widetilde{X}$ and $\widetilde{Y}$. *Hint:* It may be helpful to use the fact that a map $f : A \to B$ is a homotopy equivalence if and only if there exist maps $g, h : B \to A$ such that $f \circ g$ and $h \circ f$ are both homotopy equivalences.

6. Consider $SO(3)$, the special orthogonal group consisting of 3×3 matrices A such that $A^t \cdot A = A \cdot A^t = I$.

a) Considering the matrix entries as the coordinates on $\mathbb{R}^9$, show that $SO(3)$ is a smooth submanifold of $\mathbb{R}^9$. What is its dimension?

b) Identify the tangent space to $SO(3)$ at the identity matrix. Your answer should consist of a precise description of the set of 3×3-matrices which forms this tangent space.

c) What is the tangent space to $SO(3)$ at any given point $A \in SO(3)$?

7. Let M be a compact oriented $(n+1)$-dimensional manifold with boundary, and let $f : \partial M \to X$ be a smooth map to a closed, compact, oriented n-manifold X. Prove that if f extends to a map $M \to X$ then $\deg(f) = 0$.

8. Consider the graph $G \subset \mathbb{R}^2$ of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = |x|$.

a) Describe a smooth manifold structure on G.

b) Does there exist a smooth manifold structure on G making it a smooth submanifold of $\mathbb{R}^2$? (A rigorous justification is required to get full credit!)

9. a) Let M be a smooth manifold and A and B two smooth submanifolds of M. Give a precise definition of the statement that A and B *intersect transversely.*

b) Let A be the unit sphere in $\mathbb{R}^3$, and let $B \subset \mathbb{R}^3$ be defined by the parametrization

$$x = s, \ y = t, \ z = s^2 + t^2,$$

where $(s, t) \in \mathbb{R}^2$. Do A and B intersect transversely? Justify your answer.