Name: ____________________________

Instructions: This is a four hour exam and 'closed book’. There are eight problems.

1. (a) Suppose that \( t \in \mathbb{R} \) is a regular value of a smooth map \( f : \mathbb{R}^n \to \mathbb{R} \), and let \( M = f^{-1}(t) \). Explain why \( M \) has a nowhere vanishing normal vector field.

(b) If \( f(x, y, z) = x^2 + y^2 + z^2 \), check that the hypothesis of part (a) holds when \( t = 1 \), and then draw a picture illustrating the conclusion.
2. Rigorously prove that the Möbius band is non-orientable.
3. (a) Let $M$ and $N$ be smooth connected closed (= compact without boundary) manifolds of the same dimension. Show that a submersion $f : M \to N$ will then be a finite sheeted covering map. (a submersion = a map whose differential is surjective at each point.)

(b) Explain why if $M$ is a connected closed surface, and $f : M \to S^2$ is a submersion, then $f$ must, in fact, be a diffeomorphism.

(c) Explain why if $M$ is a connected closed surface, and $f : M \to S^1 \times S^1$ is a submersion, then $M$ must be $S^1 \times S^1$. 
4. Let $S^2 \xrightarrow{p_1} S^2 \vee S^2 \xrightarrow{p_2} S^2$ be the two ‘projection maps’: the other sphere is collapsed to the basepoint. Then say that a map $f : S^2 \to S^2 \vee S^2$ has type $(m, n)$ if the degree of $p_1 \circ f$ is $m$ and the degree of $p_2 \circ f$ is $n$. Let $X_f = (S^2 \vee S^2) \cup_f D^3$.

(a) Compute the homology groups of $X_f$ if $f$ has type $(4, 6)$, describing the homology groups as direct sums of cyclic groups, as usual.

(b) More generally, describe the homology groups of $X_f$ if $f$ has type $(m, n)$. 

5. Suppose that $X$ is the union of open sets $X_1$ and $X_2$, and $Y$ is the union of open sets $Y_1$ and $Y_2$. Let $f : X \to Y$ be a map that restricts to maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, and thus also $f_{12} : X_1 \cap X_2 \to Y_1 \cap Y_2$.

Prove that, if $f_1$, $f_2$ and $f_{12}$ all induce isomorphisms in homology, then $f_* : H_*(X) \to H_*(Y)$ will also be an isomorphism.
6. Suppose $p : \tilde{Y} \to Y$ is a double cover. If $X$ is a space such that $H_1(X)$ is a finite group of odd order, show that any map $f : X \to Y$ lifts through $p$: there exists $\tilde{f} : X \to \tilde{Y}$ such that $f = p \circ \tilde{f}$. (You can assume that $X$ is locally ‘friendly’.)
7. Let $M_2(\mathbb{R})$ be the vector space of all $2 \times 2$ real matrices, and let $f : M_2(\mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det(A)$. The differential of $f$ at $A \in M_2(\mathbb{R})$ is a linear map $d_A f : M_2(\mathbb{R}) \to \mathbb{R}$.

(a) Compute $d_A f(A)$.

(b) Show that $SL_2(\mathbb{R})$, the group of $2 \times 2$ real matrices with determinant 1, is a smooth submanifold of $M_2(\mathbb{R})$.

(c) Show that $T_I SL_2(\mathbb{R})$, the tangent space of $SL_2(\mathbb{R})$ at the identity matrix $I$, is the subspace of $M_2(\mathbb{R})$ consisting of matrices with trace equal to 0.
8. Recall that the Brower Fixed Point Theorem says that every continuous self map of the closed \( n \)-ball \( D^n \) has a fixed point.

(a) Prove the theorem using homology.

(b) Prove the theorem using the methods of differential topology methods. (Step 1: If a continuous \( f \) had no fixed points, a nearby smooth function would also have no fixed points.)
Other ideas for problems:

Extra 1. (a) Describe a smooth atlas for \( \mathbb{R}P^n \).

(b) Describe a C.W. complex structure for \( \mathbb{R}P^n \).

Other problems suggested by Slava . . .

Extra 2. View \( \mathbb{R}P^n \) as the space of lines through the origin in \( \mathbb{R}^{n+1} \). Show that, given a continuous map \( f: \mathbb{R}P^n \to \mathbb{R}^{n+1} - \{0\} \), there exists \( L \in \mathbb{R}P^n \) such that the vector \( f(L) \) is orthogonal to the line \( L \). (hmm ... we need \( n > 0 \).)

Nick’s comments . . . Alternative (and equivalent) Show that, for \( n > 0 \), there is no continuous map \( f: \mathbb{R}P^n \to \mathbb{R}^{n+1} - \{0\} \) such that \( f(L) \in L \) for all lines \( L \).

Remark This seems to have a simple proof that doesn’t involve any diff or alg topology: From such an \( f \) that shouldn’t exist, one gets \( g: S^n \to S^n \) such that (i) \( g(x) \) is either \( x \) or \( -x \) for all \( x \), and (ii) \( g(x) = g(-x) \) for all \( x \). Since \( S^n \) is connected, (i) implies that \( g \) is either the identity or the antipodal map, and neither of these satisfy (ii).

Extra 3. Consider a smooth map \( f: S^3 \to S^2 \) and let \( x, y \in S^2 \) be two regular values.

(a) Explain how orientations on the spheres \( S^2, S^3 \) induce an orientation of the 1-dimensional submanifolds \( f^{-1}(x), f^{-1}(y) \subset S^3 \). Using these orientations, state a definition of the linking number \( \text{lk}(f^{-1}(x), f^{-1}(y)) \).

(b) Suppose \( f \) is smoothly homotopic to a constant map. Show that in this case \( \text{lk}(f^{-1}(x), f^{-1}(y)) = 0 \). [Hint: you may use the fact that the linking number may be computed as the intersection number of surfaces bounded by the 1-manifolds in \( D^4 \).]

Question from Nick . . . What sort of answer would one want in part (a)?