Name: ____________________________

**Instructions:** This is a four hour exam and ‘closed book’. There are eight problems.
1. (a) Let \( T \subset \mathbb{R}^5 \) be a closed subspace homeomorphic to \( \mathbb{R}^2 \). Explain why \( T \) will be a retract of \( \mathbb{R}^5 \).

(b) View \( S^n \) as \( \mathbb{R}^n \cup \{\infty\} \), so that the open subsets of \( S^n \) containing \( \infty \) are precisely the complements of compact subsets of \( \mathbb{R}^n \). Recall that a continuous function \( f : \mathbb{R}^m \to \mathbb{R}^n \) is called proper if \( f^{-1}(C) \) is compact in \( \mathbb{R}^m \) whenever \( C \) is compact in \( \mathbb{R}^n \). Show that such a proper map extends uniquely to a continuous function \( \bar{f} : S^m \to S^n \).

(c) With \( T \) as in part (a), check that the inclusion \( i : T \hookrightarrow \mathbb{R}^5 \) is proper. By contrast, show that no retraction \( r : \mathbb{R}^5 \to T \) can be proper. (Hint: start by using part (b).)
2. Let $\mathbb{R}^\infty$ denote the union $\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \ldots$, with the union topology, i.e. $U \subset \mathbb{R}^\infty$ is open iff $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all $n$. Let $\mathbb{R}^\omega$ denote the product of a countable number of copies of $\mathbb{R}$, with the product topology. Check that the evident set theoretic inclusion $i : \mathbb{R}^\infty \to \mathbb{R}^\omega$ is continuous, but is not a homeomorphism onto its image.
3. (a) Describe a connected double cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$. (There is more than one correct answer.)

(b) What are the homology groups of your double cover?

(c) What is the fundamental group of your double cover?
4. Let $M$ be the compact surface with boundary circle $C$ as pictured:

(a) Explain why $M$ is homotopy equivalent to a figure eight. (Hint: $M$ is the torus with a disk removed, and the torus is often represented as a square with opposite edges identified.)

(b) Explain why the inclusion $i : C \hookrightarrow M$ induces the zero homomorphism from $H_1(C)$ to $H_1(M)$.

(c) By contrast, explain why $i$ is not null homotopic.
5. Suppose given a commutative diagram of abelian groups

\[
\begin{array}{ccc}
0 & 0 & \\
\downarrow & \downarrow & \\
B_1 & \beta & B_2 \\
\downarrow & \downarrow & \\
0 & A_1 & C & D & 0 \\
\alpha & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & A_2 & E & F & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & \\
\end{array}
\]

with exact rows and columns. Show that there are isomorphisms

\[
\ker \alpha \simeq \ker \beta \quad \text{and} \quad \coker \alpha \simeq \coker \beta.
\]
6. Recall that an $n$-dimensional manifold is a Hausdorff topological space $M$ that can be covered by open sets homeomorphic to open sets in $\mathbb{R}^n$. Prove that a compact $n$-dimensional manifold can be embedded in (i.e. is homeomorphic to a subset of) $\mathbb{R}^N$ for large enough $N$. (Hint: use a partition of unity associated to a finite open cover $U_1, \ldots, U_k$ of $M$ equipped with embeddings $f_i : U_i \rightarrow \mathbb{R}^n$.)
Let \( C \subset \mathbb{R}^3 \) be the union of the \( x \)-axis and the \( y \)-axis. Compute \( H_\ast(\mathbb{R}^3 - C) \). (Hint: note that \( \mathbb{R}^3 - C = (\mathbb{R}^3 - x\text{-axis}) \cap (\mathbb{R}^3 - y\text{-axis}) \).)
8. Let $X$ be a Hausdorff space, and $f : X \to X$ a continuous function such that
   - $f(x) \neq x$ for all $x \in X$, and
   - $f \circ f$ is the identity.

(a) Show that every $x \in X$ has an open neighborhood $W_x$ satisfying $f(W_x) \cap W_x = \emptyset$.

(b) Let $\bar{X} = X/(x \sim f(x))$, with the quotient space topology. Show that the quotient map $q : X \to \bar{X}$ is a covering map.