Topology general exam

Solve 7 of the following 8 problems.

1. (a) Describe the fundamental group and universal cover of the torus $S^1 \times S^1$.
   (b) Prove that any smooth map from the two-sphere $S^2$ to the torus has $(\text{mod } 2)$ degree equal to zero.

2. Let $M$ be a 2-dimensional submanifold of $\mathbb{R}^3$, and let $d : M \to \mathbb{R}$ be the distance to the origin. Suppose the origin lies in the complement of $M$. Show that the critical points of $d$ are precisely the points of $M$ where $M$ is tangent to some sphere centered at the origin.

3. Let $(X, \mathcal{T})$ be a topological space. Let $Y$ be the union of $X$ and a point $p$ (not in $X$). Let $\mathcal{S}$ be the collection of subsets of $Y$ given by
   (1) if $U \subset Y$ and $p \notin U$, then $U \in \mathcal{S}$ if and only if $U$ is open in $X$,
   (2) if $U \subset Y$ and $p \in U$, then $U \in \mathcal{S}$ if and only if $Y \setminus U$ is closed and compact in $X$.
   Show that:
   (1) $\mathcal{S}$ is a topology for $Y$.
   (2) $Y$ is compact.
   (3) $X$ is an open subset of $Y$ and the topology induced on $X$ from $\mathcal{S}$ is $\mathcal{T}$.
   (4) $X$ is dense in $Y$ if and only if $(X, \mathcal{T})$ is not compact.
   (5) $Y$ is a $T_1$ space if and only if $X$ is $T_1$.
   (6) $Y$ is Hausdorff if and only if $X$ is Hausdorff and locally compact.
   (7) If $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_1^n x_i^2 < 1\}$ with the standard topology, prove that $Y$ is homeomorphic to $S^n$.

4. (a) Let $X$ be a topological space. Let $U_1$, $U_2$ be dense open subsets. Prove that their intersection is a dense subset of $X$.
   (b) Let $X$ be a compact Hausdorff space. Let $A$ be a subset of $X$ and let $U$ be an open subset of $X$ such that closure$(A) \subset U$. Prove that there exists an open set $W$ such that closure$(A) \subset W$ and closure$(W) \subset U$.
   (c) Let $X$ be a compact Hausdorff space. Let $\{U_n\}$ be a countable collection of dense open subsets of $X$. Prove that their intersection is a dense subset of $X$.
   (d) Give an example of a compact space $X$, and a countable collection of dense open subsets of $X$, such that the intersection is not dense in $X$.

5. (a) Describe the universal cover of the figure eight $S^1 \vee S^1$.
   (b) Describe a non-trivial two-fold covering space of $S^1 \vee S^1$. How many different two-fold covering spaces of $S^1 \vee S^1$ are there?
6. Show that the special linear group $SL(n, \mathbb{R})$ is a smooth manifold.

7. Let $M$ be a non-empty smooth $n$-dimensional manifold, and $f: M \rightarrow \mathbb{R}$ be a smooth map.
   (a) Show that if $M$ is compact then there are elements $r$ in $\mathbb{R}$ which are not regular values.
   (b) If $S^n = \{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum_0^n x_i^2 = 1 \}$ and $f: S^n \rightarrow \mathbb{R}$ is given by $f(x_0, \ldots, x_n) = x_0^2$, what are the regular values of $f$? Show that $f^{-1}(r)$ is a submanifold of $S^n$ for all values of $r$.

8. Prove that any smooth map $f: D^n \rightarrow D^n$ has a fixed point, for any $n \geq 1$. 