Topology general exam

Solve 7 of the following 8 problems.

1. (a) Describe the fundamental group and universal cover of the torus $S^1 \times S^1$.

(b) Prove that any smooth map from the two-sphere S^2 to the torus has (mod 2) degree equal to zero.

2. Let M be a 2-dimensional submanifold of $\mathbb{R}^3$, and let $d\colon M \longrightarrow \mathbb{R}$ be the distance to the origin. Suppose the origin lies in the complement of M. Show that the critical points of d are precisely the points of M where M is tangent to some sphere centered at the origin.

3. Let $(X, \mathcal{T})$ be a topological space. Let Y be the union of X and a point p (not in X). Let $\mathcal{S}$ be the collection of subsets of Y given by

(1) if $U \subset Y$ and $p \notin U$, then $U \in \mathcal{S}$ if and only if U is open in X,

(2) if $U \subset Y$ and $p \in U$, then $U \in \mathcal{S}$ if and only if $Y \setminus U$ is closed and compact in X.

Show that:
(1) $\mathcal{S}$ is a topology for Y.
(2) Y is compact.
(3) X is an open subset of Y and the topology induced on X from $\mathcal{S}$ is $\mathcal{T}$.
(4) X is dense in Y if and only if $(X, \mathcal{T})$ is not compact.
(5) Y is a T_1 space if and only if X is T_1.
(6) Y is Hausdorff if and only if X is Hausdorff and locally compact.
(7) If $X = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, \sum_1^n x_i^2 < 1\}$ with the standard topology, prove that Y is homeomorphic to S^n.

4. (a) Let X be a topological space. Let U_1, U_2 be dense open subsets. Prove that their intersection is a dense subset of X.

(b) Let X be a compact Hausdorff space. Let A be a subset of X and let U be an open subset of X such that closure$(A) \subset U$. Prove that there exists an open set W such that closure$(A) \subset W$ and closure$(W) \subset U$.

(c) Let X be a compact Hausdorff space. Let $\{U_n\}$ be a countable collection of dense open subsets of X. Prove that their intersection is a dense subset of X.

(d) Give an example of a compact space X, and a countable collection of dense open subsets of X, such that the intersection is not dense in X.

5. (a) Describe the universal cover of the figure eight $S^1 \vee S^1$.

(b) Describe a non-trivial two-fold covering space of $S^1 \vee S^1$. How many different two-fold covering spaces of $S^1 \vee S^1$ are there?

6. Show that the special linear group $SL(n,\mathbb{R})$ is a smooth manifold.

7. Let M be a non-empty smooth n-dimensional manifold, and $f\colon M \longrightarrow \mathbb{R}$ be a smooth map.

(a) Show that if M is compact then there are elements r in $\mathbb{R}$ which are not regular values.

(b) If $S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \,|\, \sum_0^n x_i^2 = 1\}$ and $f : S^n \longrightarrow \mathbb{R}$ is given by $f(x_0, \ldots, x_n) = x_0^2$, what are the regular values of f? Show that $f^{-1}(r)$ is a submanifold of S^n for all values of r.

8. Prove that any smooth map $f\colon D^n \longrightarrow D^n$ has a fixed point, for any $n \geq 1$.