Topology General Exam **Fall 2002**

Guidelines This is three hours, and "closed book".

1. Let $S_1, S_2, \ldots$ be a sequence of finite sets each having at least two elements. Each S_n is a topological space with the discrete topology. Let

$$X = \prod_{n=0}^{\infty} S_n$$

be given the product topology.

(a) Is X discrete? Hausdorff? Compact? Connected? Normal? Metrizable?

(b) Describe the path connected components of X.

2. Recall that a topological group G is a group that is also a topological space, such that the functions

$$m : G \times G \longrightarrow G \text{ and } i : G \longrightarrow G$$

defined by $m(a, b) = ab$ and $i(a) = a^{-1}$ are continuous.

(a) Prove the useful lemma: if G is a topological group, and U is an open neighborhood of the unit e, then e has another open neighborhood W such that

$$a, b \in W \Rightarrow a^{-1}b \in U.$$

(b) Suppose the topological group G is connected, and $H \subset G$ is a *discrete* subgroup, i.e. a subgroup which is discrete with the subspace topology. Let $p : G \to G/H$ be the projection on the space of cosets, i.e. $p(g) = gH$, and give G/H the quotient topology. Show that p is a covering space map.

(c) With G and H as in (b), how are the fundamental groups $\pi_1(G, e)$ and $\pi_1(G/H, eH)$ related?

3. As a space covered by $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$ has the structure of a smooth manifold. The circle $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is a smooth submanifold of $\mathbb{R}^2$. Prove the intuitively obvious fact: $\mathbb{R}/\mathbb{Z}$ is diffeomorphic to S^1.

4. Let R be the figure eight space:

(a) Explain (e.g. with convincing pictures) why R is a retract of the genus 2 surface:

(b) In contrast, prove that R is *not* a retract of the torus $S^1 \times S^1$:

5. Recall that $\mathbb{R}P^2 = S^2/(\sim)$, where $(x, y, z) \sim (-x, -y, -z)$ defines the equivalence relation. Write down an explicit smooth atlas for $\mathbb{R}P^2$, exhibiting it as a 2 dimensional smooth manifold. Remark: your atlas will need at least three charts.

6. Let M be a smooth manifold of dimension n, and $f : M \to \mathbb{R}^N$ a smooth map with $N > 2n$.

(a) Let $g : TM \to \mathbb{R}^N$ be defined by $g(v) = df_x(v)$ for $v \in T_xM$. Explain why g cannot be onto.

(b) Let $v \in \mathbb{R}^N$ be chosen to *not* be in the image of g, and let $L : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be a surjective linear map satisfying $L(v) = 0$. Show that, if the original map f is an immersion, then so is $L \circ f : M \to \mathbb{R}^{N-1}$.

7. **(a)** Give the definition of a 1–form on a smooth manifold M.

(b) Show that the vector space of all 1–forms on S^1 is isomorphic to the vector space of all functions $f : S^1 \to \mathbb{R}$.