NAME: ______________________________

PLEDGE: ____

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SIGNATURE: ______________________________

To get credit for a problem, you must include all logical steps and detailed calculations. Verify or give adequate reasons for assertions that you make. Cite by name any theorems you wish to invoke. Please write only on one side of your paper.

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PART I

1. An entire function f is said to be of *exponential type* if there exist positive constants c_1 and c_2 such that
$$|f(z)| \leq c_1\, e^{c_2|z|}$$
for all z in $\mathbb{C}$. Show that f is of exponential type if and only f' is of exponential type.

2. Let f be an entire function such that
$$|f(z)| \leq 2\, e^{x} \quad \text{for all } z = x + iy \quad \text{such that} \quad |z| > 100.$$
Show that f is a multiple of the complex exponential function $e(z) = e^{z}$.

3. Let Ω be an open set containing the closed unit disc, and $\{f_n\}_{n=1}^{\infty}$ a sequence of analytic functions on Ω converging to a function f uniformly on compacts (in Ω). Suppose that the minimum of $|f|$ on the unit circle is a strictly positive number. Show that there exists a positive integer n_0 such that the functions f_n have the same number of zeros in the open unit disc for all $n \geq n_0$.

4. Show that
$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

5. Let
$$A = \{z \in \mathbb{C} : \, 0 < |z| < 1\} \quad \text{and} \quad B = \{z \in \mathbb{C} : \, 1 < |z| < 2\}.$$
Show that A and B are not conformally equivalent.

PART II

6. Let $(X, \mathcal{M})$ denote a measurable space and suppose μ, ν are two positive measures on $\mathcal{M}$ such that $\mu \le \nu$, that is, $\mu(A) \le \nu(A)$ for all $A \in \mathcal{M}$. Show that
$$\int_X f\, d\mu \le \int_X f\, d\nu$$
for all measurable $f : X \to [0, \infty]$.

7. For $p \in [1, \infty]$, let $L^p = L^p(X, \mathcal{M})$ where $(X, \mathcal{M})$ is a measurable space.

   Prove that, for $1 \le p < q < r \le \infty$, $L^p \cap L^r$ equipped with the norm $\|f\| := \|f\|_p + \|f\|_r$ is a Banach space and the inclusion map $\iota : L^p \cap L^r \to L^q$ given by $\iota(f) = f$ is continuous with respect to the norm topologies.

8. Prove that
$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n dx = 1.$$

9. Suppose that f is Lebesgue integrable on $(0, 1)$ and $g : (0, 1) \to \mathbb{R}$ is defined by
$$g(x) := \int_x^1 \frac{f(t)}{t}\, dt.$$
   Prove that g is integrable on $(0, 1)$ and
$$\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx.$$

10. Suppose H is a Hilbert space and $M_n \subset H$ is an increasing sequence of closed subspaces. Let $M := \cup_{n=1}^\infty M_n$ and $P_n = P_{M_n}$ and $P_{\overline{M}}$ be orthogonal projection onto M_n and $\overline{M}$ (the closure of M) respectively. Show that
$$\lim_{n\to\infty} P_n x = P_{\overline{M}} x$$
for all $x \in H$.

    Hint: First prove it for $x \in M^\perp$, then $x \in M$, and then $x \in \overline{M}$.