NAME: ________________________________

PLEDGE: ________________________________

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To get credit for a problem, you must show all of your reasoning and calculations.

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1. Suppose $f : [0, 1] \to \mathbb{C}$ is continuous. Find

$$\lim_{k \to \infty} \int_{0}^{1} k x^{k-1} f(x) dx.$$ 

Prove your result.
2. (a) Show that any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals $(a, b)$ (where $-\infty \leq a < b \leq \infty$).

(b) Show that the $\sigma$-algebra generated by the open subsets of $\mathbb{R}$ is the same as the $\sigma$-algebra generated by the intervals $[a, b)$ with $-\infty < a < b < \infty$. 
3. Suppose $(X, \Sigma)$ is a measurable space, and $\nu$ and $\mu$ are two measures on the $\sigma$-algebra $\Sigma$ with $\nu$ absolutely continuous with respect to $\mu$. Suppose $\nu$ is a finite measure. Show that $\nu(E) \to 0$ as $\mu(E) \to 0$. (In other words, given $\epsilon > 0$ there exists $\delta > 0$ such that if $E \in \Sigma$ with $\mu(E) < \delta$, then $\nu(E) < \epsilon$.)

[Hint: If not show $\exists \, \epsilon > 0$ and $E_j \in \Sigma$ such that $\bigcup_{j=1}^{\infty} E_j < \infty$ and $\nu(E_j) \geq \epsilon$. Proceed from there.]
4. (a) Use the Monotone Convergence Theorem and \( \int_1^t \frac{dx}{x} = \log t \) to show

\[
\lim_{n \to \infty} n \log \left( 1 + \frac{1}{n} \right) = t \quad \text{for} \quad t \geq 0.
\]

(b) Show \( \lim_{n \to \infty} \int_0^n \left( 1 + \frac{t}{n} \right)^n e^{-2t} dt = 1 \).
(c) Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ for $x > 0$. Show that

\[ \begin{aligned}
\Gamma(x) &= \lim_{n \to \infty} \int_0^n \left( 1 - \frac{t}{n} \right)^n t^{x-1} dt \\
&= \lim_{n \to \infty} n^x n!(x(x + 1) \cdots (x + n))^{-1}.
\end{aligned} \]
5. (a) State the Mean Value Property for analytic functions, then use the Cauchy Integral Formula to prove it.

(b) TRUE or FALSE: If $u$ is a harmonic function on a domain in $\mathbb{R}^2$, then $u$ has a harmonic conjugate.
(c) Find a fractional linear transformation (also called a Möbius transformation) $f$ that takes the first quadrant to the top half of the unit disk and satisfies $f(2) = i$. (You must explain some comprehensible procedure and not simply produce an $f$ out of thin air.) Under your map, what is the image of the vertical ray $\{\Re z = c > 0, \Im z > 0\}$?
6. Let $f(z)$ be a bounded analytic function in the upper half-plane that extends continuously to the real axis. If $|f(z)| \leq M$ for real $z$, show that $|f(z)| \leq M$ for all $z$ in the upper half-plane.

[Suggestion: for the top half of an arbitrary disk centered at the origin, consider an appropriate branch of the function $(z + i)^{-\varepsilon}f(z)$ for small enough $\varepsilon > 0$.]
7. Use the argument principle to determine the number of roots of \( p(z) = z^9 + 4z^5 - 3z^4 + 4z + \alpha \) in the right half-plane. The answer may depend on the value of \( \alpha \), which is assumed real.
8. Let \(a\) and \(b\) be unequal positive numbers. By integrating an appropriate branch of \(\frac{(\log z)^2}{(z+a)(z+b)}\) around a keyhole contour, find

\[
\int_0^\infty \frac{\log x}{(x+a)(x+b)} \, dx.
\]