Name: ______________________________

Pledge: ____

____

Signature: ______________________________

To get credit for a problem, you must show all of your reasoning and calculations.

Pr	**Score**
1	
2	
3	
4	
5	
6	
7	
8	

1. Suppose $f : [0,1] \to \mathbb{C}$ is continuous. Find

$$\lim_{k\to\infty} \int_0^1 k\, x^{k-1} f(x) dx.$$

   Prove your result.

2. (a) Show that any open subset of $\mathbb{R}$ is a countable disjoint union of open intervals (a, b) (where $-\infty \leq a < b \leq \infty$).

(b) Show that the σ-algebra generated by the open subsets of $\mathbb{R}$ is the same as the σ-algebra generated by the intervals $[a, b)$ with $-\infty < a < b < \infty$.

3. Suppose (X, Σ) is a measurable space, and ν and μ are two measures on the σ-algebra Σ with ν absolutely continuous with respect to μ. Suppose ν is a finite measure. Show that $\nu(E) \to 0$ as $\mu(E) \to 0$. (In other words, given $\epsilon > 0$ there exists $\delta > 0$ such that if $E \in \Sigma$ with $\mu(E) < \delta$, then $\nu(E) < \epsilon$.)

   [Hint: If not show $\exists\ \epsilon > 0$ and $E_j \in \Sigma$ such that $\bigcup_{j=1}^{\infty} E_j < \infty$ and $\nu(E_j) \geq \epsilon$. Proceed from there.]

4. (a) Use the Monotone Convergence Theorem and $\displaystyle\int_1^t \frac{dx}{x} = \log t$ to show

$$\lim_{n\to\infty} n \log\left(1+\frac{t}{n}\right) = t \text{ for } t \geq 0.$$

(b) Show $\displaystyle\lim_{n\to\infty} \int_0^n \left(1+\frac{t}{n}\right)^n e^{-2t} dt = 1.$

(c) Let $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$. Show that

$$\begin{aligned}\Gamma(x) &= \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt \\ &= \lim_{n\to\infty} n^x n!(x(x+1)\cdots(x+n))^{-1}.\end{aligned}$$

5. (a) State the Mean Value Property for analytic functions, then use the Cauchy Integral Formula to prove it.

(b) TRUE or FALSE: If u is a harmonic function on a domain in $\mathbb{R}^2$, then u has a harmonic conjugate.

(c) Find a fractional linear transformation (also called a Möbius transformation) f that takes the first quadrant to the top half of the unit disk and satisfies $f(2) = i$. (You must explain some comprehensible procedure and not simply produce an f out of thin air.) Under your map, what is the image of the vertical ray $\{\mathrm{Re}\, z = c > 0,\ \mathrm{Im}\, z > 0\}$?

6. Let $f(z)$ be a bounded analytic function in the upper half-plane that extends continuously to the real axis. If $|f(z)| \le M$ for real z, show that $|f(z)| \le M$ for all z in the upper half-plane.

   [Suggestion: for the top half of an arbitrary disk centered at the origin, consider an appropriate branch of the function $(z+i)^{-\varepsilon} f(z)$ for small enough $\varepsilon > 0$.]

7. Use the argument principle to determine the number of roots of $p(z) = z^9 + 4z^5 - 3z^4 + 4z + \alpha$ in the right half-plane. The answer may depend on the value of α, which is assumed real.

8. Let a and b be unequal positive numbers. By integrating an appropriate branch of $\frac{(\log z)^2}{(z+a)(z+b)}$ around a keyhole contour, find

$$\int_0^\infty \frac{\log x}{(x+a)(x+b)}\,dx.$$