Name: ______________________________

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Instructions. *4 hours. Each problem is worth the same (25 points). To get credit for a problem, you must carefully justify all (nontrivial) claims and show all calculations. You may use without proof anything that is proved in the texts by Folland and Bak and Newman, or other standard reference. If you do so, either refer to the theorem by name (if it has one) or give its statement; also verify explicitly all of its hypotheses. However, you may not cite a statement you are explicitly asked to prove, or facts that were given as exercises or homework.*

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1	
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Total points: 200

1. For any Borel measurable subset A of $\mathbb{R}$, define

$$\tilde{A} = \{(x, y) : x + y \in A\},$$

and define the measure $\mu(A) = m_2(\tilde{A} \cap [0, \infty)^2)$, where m_2 is Lebesgue measure on $\mathbb{R}^2$. Find $g : \mathbb{R} \to [0, \infty)$ such that for all A,

$$\mu(A) = \int_A g(x)dx.$$

2. Let (X, Σ, μ) be a measure space and suppose that A, B, A_i, B_i are members of Σ with the properties $(A_i \times B_i) \cap (A_j \times B_j) = \varnothing$ for $i \neq j$, and

$$A \times B = \bigcup_{i=1}^{\infty} A_i \times B_i.$$

Show that

$$\mu(A)\mu(B) = \sum_{i=1}^{\infty} \mu(A_i)\mu(B_i).$$

(**Note:** The above calculation is used to construct product measure on $X \times X$, so do not use product measure in your answer.)

3. Suppose $\phi(z,t) : U \times [a,b] \to \mathbb{C}$ where U is an open subset of $\mathbb{C}$. Suppose ϕ is continuous in t for fixed z, analytic in z for fixed t, and bounded on compacts of its domain. Let

$$f(z) = \int_a^b \phi(z,t)dt.$$

Show that f is analytic in U and

$$f'(z) = \int_a^b \frac{\partial}{\partial z}\phi(z,t)dt.$$

4. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $|a_0| > 1$, $n \geq 1$. Show that p has a zero outside the closed unit disk. (**Hint:** Factor p.)

5. Show that for each $\epsilon > 0, f(z) = \sin z + (z^2 + i)^{-1}$ has infinitely many zeros in $\{z : |Imz| < \epsilon\}$.

6. Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function and, for $n = 1, 2, \ldots$, define $f_0 \equiv 1$ and
$$f_n(t) = 1 + \int_0^t G(f_{n-1}(s))ds, \qquad t \in [-1, 1].$$
Show that $\{f_n\}$ has a uniformly convergent subsequence.

7. Compute $\int_0^\pi \tan(\theta + ia)d\theta$; $a \in \mathbb{R}, a \neq 0$.

8. (a) State and prove the dominated convergence theorem.

   (b) Let $\{f_n\}$ be a sequence of measurable functions on (X, Σ, μ) that converges pointwise a.e. to f. Suppose $\{g_n\}$ is a sequence of integrable functions on X that converges pointwise a.e. and in L^1 to g, such that $|f_n| \leq g_n$ for all n. Show that

$$\lim_{n\to\infty} \int_X f_n = \int_X f.$$

   (**Hint:** Rework your proof to part (a))