GENERAL EXAM - ANALYSIS
January, 2011

Closed book, closed notes. Please pledge. In each problem, justify all assertions, show calculations, and identify those theorems which you invoke in your arguments.

1. Let $\{f_n\}$ be a sequence of real-valued continuous functions on $[0, 1]$ which is monotone non-increasing $f_{n+1}(x) \leq f_n(x)$ for all $x \in [0, 1]$ and such that

$$\lim_{n\to\infty} f_n(x) = 0.$$

a) Prove that the convergence is uniform.

b) Show that, if instead $\{f_n\}$ is again a monotone sequence of continuous converging pointwise to a function f which is however not continuous, then the convergence is not uniform.

2. Let $\{f_n\}$ be a sequence of real-valued Borel measurable functions on $\mathbb{R}$.

a) Show that

$$f(x) \equiv \sup_n f_n(x)$$

and

$$g(x) \equiv \limsup f_n(x)$$

are measuable.

b) Define the set K,

$$K = \{x : \; f_n(x) \in (0, 1) \text{ for infinitely many n's}\}.$$

Show that K is a Borel measurable set.

3. Let m be Lebesgue measure, and suppose that $f(x, y)$ is a Lebesgue measurable non-negative function on the plane $\mathbb{R}^2$ such that

$$F(\lambda, y) = m\{x : \; f(x, y) \geq \lambda\}$$

satisfies

$$\int_0^\infty \int_{\mathbb{R}} \lambda^r F(\lambda, y) dy \, d\lambda < \infty$$

for some $r \geq 0$.

Let

$$G(\lambda, x) = m\{y : \; f(x, y) \geq \lambda\}.$$

a) Show that

$$\int_0^\infty \int_{\mathbb{R}} \lambda^r G(\lambda, x) dx \, d\lambda < \infty$$

b) Show that $f \in L^{r+1}(\mathbb{R}^2, dxdy)$, i.e.,

$$\int_{\mathbb{R}^2} f^{r+1}(x, y) dx \, dy < \infty.$$

Show also that

$$m \times m\{(x,y) \in \mathbb{R}^2 : f(x,y) \geq \lambda\} \leq \frac{c}{\lambda^{r+1}},$$

with $m \times m$ Lebesgue measure on the plane and with c a finite constant.

4. Let $\{f_n\}$ be the sequence of functions defined on $[0, 2\pi]$ with

$$f_n(x) = \sum_{k=1}^{n} \frac{e^{ikx}}{k^{3/4}}.$$

a) Show that $\{f_n\}$ converges in an $L^2([0,2\pi], dx)$-sense, $n \to \infty$.

b) Show that $\{f_n\}$ converges in an $L^1[(0,2\pi], dx)$-sense, $n \to \infty$.

5. Using residue methods, find

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}} dx$$

by considering

$$\int_{\Gamma} \frac{e^{iz}}{e^z + e^{-z}} dz$$

where Γ is the rectangle as shown with a suitably chosen value for the height.

6. Suppose that f is analytic in an open connected set Ω, and that all values of f on Ω lie in the disk of radius $M > 0$ centered at 0. Prove that

$$(*) \quad |f'(z)| \leq \frac{M}{d(z)}$$

for all $z \in \Omega$, where $d(z)$ is the distance from z to the boundary of Ω. Then show that $(*)$ can be used to prove Liouville's theorem.

7. Suppose f is analytic in a set containing the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ with $f(-\log 2) = 0$ and $|f(z)| \leq |e^z|$ for all z with $|z| = 1$. How large can $|f(\log 2)|$ be? (Here, $\log z$ denotes the principal branch of the logarithm.)

8. a)Find the image of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ under the mapping

$$g(z) = \frac{z+1}{1-z}.$$

b) Find the image of all straight lines through the point $z = 1$ under this mapping.

c) Show that the function

$$f(z) = e^{-g(z)}$$

is bounded on the unit disk. Determine the limit of $f(z)$ as $z \to 1$ along any line segment lying within the unit disk. What is the limit as $z \to 1$ along the unit circle $\{z : |z| = 1\}$?