1. Let \( \{f_n\} \) be a sequence of real-valued continuous functions on \([0, 1]\) which is monotone non-increasing \( f_{n+1}(x) \leq f_n(x) \) for all \( x \in [0, 1] \) and such that
\[
\lim_{n \to \infty} f_n(x) = 0.
\]
   a) Prove that the convergence is uniform.

   b) Show that, if instead \( \{f_n\} \) is again a monotone sequence of continuous converging pointwise to a function \( f \) which is however not continuous, then the convergence is not uniform.

2. Let \( \{f_n\} \) be a sequence of real-valued Borel measurable functions on \( \mathbb{R} \).
   a) Show that
   \[
f(x) \equiv \sup_n f_n(x)
   \]
   and
   \[
g(x) \equiv \limsup_n f_n(x)
   \]
   are measurable.

   b) Define the set \( K \),
   \[
   K = \{ x : f_n(x) \in (0, 1) \text{ for infinitely many } n \text{'s} \}.
   \]
   Show that \( K \) is a Borel measurable set.

3. Let \( m \) be Lebesgue measure, and suppose that \( f(x, y) \) is a Lebesgue measurable non-negative function on the plane \( \mathbb{R}^2 \) such that
   \[
   F(\lambda, y) = m\{ x : f(x, y) \geq \lambda \}
   \]
   satisfies
   \[
   \int_0^\infty \int_{\mathbb{R}} \lambda^r F(\lambda, y) dy d\lambda < \infty
   \]
   for some \( r \geq 0 \).
   Let
   \[
   G(\lambda, x) = m\{ y : f(x, y) \geq \lambda \}.
   \]
   a) Show that
   \[
   \int_0^\infty \int_{\mathbb{R}} \lambda^r G(\lambda, x) dx d\lambda < \infty
   \]
   b) Show that \( f \in L^{r+1}(\mathbb{R}^2, dx dy) \), i.e.,
   \[
   \int_{\mathbb{R}^2} f^{r+1}(x, y) dx dy < \infty.
   \]
Show also that
\[
m \times m \{ (x, y) \in \mathbb{R}^2 : f(x, y) \geq \lambda \} \leq \frac{c}{\lambda^{r+1}},
\]
with \( m \times m \) Lebesgue measure on the plane and with \( c \) a finite constant.

4. Let \( \{ f_n \} \) be the sequence of functions defined on \([0, 2\pi]\) with
\[
f_n(x) = \sum_{k=1}^{n} \frac{e^{ikx}}{k^{3/4}}.
\]
a) Show that \( \{ f_n \} \) converges in an \( L^2([0, 2\pi], dx) \)-sense, \( n \to \infty \).

b) Show that \( \{ f_n \} \) converges in an \( L^1([0, 2\pi], dx) \)-sense, \( n \to \infty \).

5. Using residue methods, find
\[
\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}} dx
\]
by considering
\[
\int_{\Gamma} \frac{e^{iz}}{e^z + e^{-z}} dz
\]
where \( \Gamma \) is the rectangle as shown with a suitably chosen value for the height.

6. Suppose that \( f \) is analytic in an open connected set \( \Omega \), and that all values of \( f \) on \( \Omega \) lie in the disk of radius \( M > 0 \) centered at 0. Prove that
\[
(*) \ |f'(z)| \leq \frac{M}{d(z)}
\]
for all \( z \in \Omega \), where \( d(z) \) is the distance from \( z \) to the boundary of \( \Omega \). Then show that \( (*) \) can be used to prove Liouville’s theorem.

7. Suppose \( f \) is analytic in a set containing the closed unit disk \( \overline{D} = \{ z : |z| \leq 1 \} \) with \( f(-\log 2) = 0 \) and \( |f(z)| \leq |e^z| \) for all \( z \) with \( |z| = 1 \). How large can \( |f(\log 2)| \) be? (Here, \( \log z \) denotes the principal branch of the logarithm.)

8. a) Find the image of the unit disk \( D = \{ z : |z| < 1 \} \) under the mapping
\[
g(z) = \frac{z + 1}{1 - z}.
\]

b) Find the image of all straight lines through the point \( z = 1 \) under this mapping.

c) Show that the function
\[
f(z) = e^{-g(z)}
\]
is bounded on the unit disk. Determine the limit of \( f(z) \) as \( z \to 1 \) along any line segment lying within the unit disk. What is the limit as \( z \to 1 \) along the unit circle \( \{ z : |z| = 1 \} \)?