# GENERAL EXAM - ANALYSIS

January 2010

Closed book, closed notes. Please pledge. In each problem, justify all assertions, show calculations, and identify those theorems which you invoke in your arguments.

1. Let $w_1, w_2, \cdots, w_n$ be $n$ points on the unit circle $\mathbb{T} \equiv \{z \in \mathbb{C} : |z| = 1\}$. Show that there exists a point $z \in \mathbb{T}$ so that the product of the distances from $z$ to $w_j$ (i.e., $\prod_{j=1}^{n} |z - w_j|$) is at least 1. Then show there there is a point $v \in \mathbb{T}$ so that the product of the distances from $v$ to the points $w_j$ is exactly 1.

2. Suppose that $f$ is analytic in an open set containing the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$, except for a simple pole at $z_0$ where $|z_0| = 1$. Show that if
$$\sum_{n=0}^{\infty} a_n z^n$$
is the power series for $f$ in $\mathbb{D}$, then
$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

3. Suppose that $f$ and $g$ are analytic in an open set containing the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Suppose that $f$ has a simple zero at $z = 0$ and no other zero in $\overline{\mathbb{D}}$. Set
$$f_\epsilon(z) = f(z) + \epsilon g(z).$$
Show that if $\epsilon$ is sufficiently small, then $f_\epsilon$ has a unique zero in $\overline{\mathbb{D}}$.

4. Evaluate
$$\int_0^\infty \frac{dx}{1 + x^n}$$
for $n = 2, 3, 4, \ldots$ by integrating over the boundary of an appropriately chosen "pie-shaped" region.

5. Let $\mathcal{B}(\mathbb{R}^1)$ be the collection of Borel sets on the real line and $\mathcal{B}(\mathbb{R}^2)$ those of the plane.

   (a) Show that sections of Borel sets in the plane, e.g., sets of the form $B_y = \{x \in \mathbb{R} : (x, y) \in \mathcal{B}\}$ with $B \in \mathcal{B}(\mathbb{R}^2)$ are in $\mathcal{B}(\mathbb{R}^1)$.

   (b) Suppose that $f(x, y)$ is a Borel measurable function on the plane, $\mathbb{R}^2$. Show that for fixed $y$, $f_y(x) = f(x, y)$ is a Borel function on $\mathbb{R}$.

6. Let $0 < a < 1 < b$ be real constants. Define the function $S(x)$ on $[0, \infty)$ by the equation

$$S(x) = \sum_{n=1}^{\infty} a^n \chi_{[0,b^n]}(x)$$

where $\chi_{[0,b^n]}(x)$ is the characteristic function for $[0, b^n]$, equal to 1 on this interval and zero otherwise.

(a) Show that $S(x) \in L^p([0, \infty), dx)$,i.e., is $L^p$-integrable (with Lebesgue measure), provided $ab^{1/p} < 1$.

(b) Show that

$$S(x) \leq \frac{x^{(\ln a / \ln b)}}{1 - a}.$$

7. Let $\{\phi_n\}$ be a complete orthonormal set of functions in the real Hilbert space $L^2([0, 1])$ with inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx.$$

For $0 \leq a \leq 1$, set

$$h(x) = \sum_{n=1}^{\infty} \phi_n(x) \int_0^a \phi_n(y)\, dy.$$

(a) Show that the series converges in an $L^2$-sense to an $L^2$ function.

(b) Show that for any $b$, $0 \leq b \leq 1$,

$$\int_0^b h(x)\, dx = \sum_{n=1}^{\infty} \int_0^b \phi_n(x)\, dx \int_0^a \phi_n(y)\, dy.$$

(c) Show that

$$\int_0^b h(x)\, dx = min(a, b).$$