GENERAL EXAM - ANALYSIS

August, 2010

Closed book, closed notes. Please pledge. In each problem, justify all assertions, show calculations, and identify those theorems which you invoke in your arguments.

1. Let $f(x)$ be a real, normalized function in $L^2(\mathbf{R})$,

$$\int_{\mathbf{R}} |f(x)|^2\, dx = 1.$$

Show that (m is Lebesgue measure):

(a) For $c > 0$,

$$m\{x : |f(x)| \geq c\} \leq \frac{1}{c^2}.$$

(b) For $c > 0$ and $r > -1/2$,

$$\left| \int_{[0,c]} x^r f(x)\, dx \right| \leq \frac{c^{\frac{2r+1}{2}}}{\sqrt{2r+1}}.$$

(c) Suppose that additionally,

$$\int_{\mathbf{R}} x^2 |f(x)|^2\, dx < \infty.$$

Show that then

$$\int_{\mathbf{R}} |f(x)|\, dx < \infty.$$

2. Let $0 < r < 1$ and let

$$S_N(x) = \sum_{n\geq 1}^{N} \frac{\cos(nx)}{n^{1+r}}$$

$$S(x) = \sum_{n\geq 1}^{\infty} \frac{\cos(nx)}{n^{1+r}}.$$

(a) Show that there is a constant c_1 such that

$$|S_N(x) - S_N(y)| \leq c_1 |x-y| N^{1-r}.$$

(b) Show that there is another constant c_2 such that

$$|S(x) - S_N(x)| \leq c_2 N^{-r}.$$

(c) By suitable choice of N depending on x, y, show that $S(x)$ is Hölder continuous with index r, i.e., there is a finite c_3 such that

$$|S(x) - S(y)| \leq c_3 |x-y|^r.$$

3. Let (X, μ) be a measure space, and let $\{f_n\}$ be a sequence of real-valued integrable functions on X which converges to the function f in an L^1-sense. Suppose that moreover,

$$\sum_n \int_X |f_n(x) - f(x)| \, d\mu(x) < \infty.$$

Show that $\{f_n\}$ converges pointwise a.e. to f. To do so, consider

$$\mu(\cup_{m \geq N} \{x : |f_m(x) - f(x)| \geq \epsilon\}).$$

4. Let (X, μ) be a finite measure space, and let $\{f_n\}$ be a sequence of real-valued measurable functions converging pointwise a.e. to a measurable function f. We say that the sequence $\{f_n\}$ has *uniformly absolutely continuous integrals* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f_n| d\mu < \epsilon$$

for all n whenever E is a measurable set with $\mu(E) < \delta$. Show that in this case, $f_n \to f$ in the norm of $L^1(\mu)$.

5. Show that for every $\epsilon > 0$, the function

$$f(z) = \sin z + \frac{1}{z - i}$$

has infinitely many zeros in the set $\{z : |\text{Im } z| < \epsilon\}$.

6. For n an even integer greater than or equal to 4, compute

$$\int_{-\infty}^{\infty} \frac{x^2}{x^n + 1} \, dx.$$

Show all estimates carefully. Your answer should be a "clearly real" number.

7. Suppose f is meromorphic (analytic except for poles) in $\mathbb{C}$. Show that if

$$\int_\gamma [p(z)]^2 f(z) dz = 0$$

for every polynomial $p(z)$ and every piecewise smooth closed curve γ not passing through a pole of f, then f is entire. Hint: First show f is entire under the assumption $\int_\gamma p(z) f(z) dz = 0$ for all such p and γ.

8. Suppose f is entire and $|f(z)| \leq |z|^2$ for all z. Find all possibilities for f.