Analysis General Exam, January 2009

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded real function.

(a) Fix $\delta > 0$, and for every $\alpha \in \mathbb{R}$ consider the set

$$U_\alpha := \{x \in \mathbb{R} : \sup\{f(y) \,:\, y \in \mathbb{R},\, |y - x| < \delta\} > \alpha\}\,.$$

Show that for every $\alpha \in \mathbb{R}$ the set U_α is open.

(b) Let $u : \mathbb{R} \to \mathbb{R}$ be given by

$$u(x) := \lim_{\delta \to 0} \left(\sup\{f(y) \,:\, y \in \mathbb{R},\, |y - x| < \delta\}\right)$$

Show that the function u is well-defined and Lebesgue measurable.

Similarly, define

$$l(x) := \lim_{\delta \to 0} \left(\inf\{f(y) \,:\, y \in \mathbb{R},\, |y - x| < \delta\}\right)$$

which, by the same arguments, will also be well-defined and Lebesgue measurable.

(c) Show that the set of points where the function f is continuous is measurable.

Problem 2. Let $f : [0,1] \to \mathbb{R}$ be a bounded, Lebesgue measurable function such that

$$\int_{[0,1]} f(x) x^k \, dx = \frac{1}{k+4}, \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

Show that $f(x) = x^3$ a.e. on $[0,1]$.

Problem 3. Let F be a Lebesgue measurable subset of $[0,2] \times [0,2]$. For each $x \in [0,2]$ set $F^x := \{y \in [0,2] \,:\, (x,y) \in F\}$ and for each $y \in [0,2]$ let $F_y := \{x \in [0,2] \,:\, (x,y) \in F\}$. Show that if $|F^x| \le \frac{1}{8}$ then

$$|\{y \in [0,2] \,:\, |F_y| \ge 1\}| \le \tfrac{1}{4}.$$

Here, if E is a Lebesgue measurable subset of $\mathbb{R}$, we denote by $|E|$ its Lebesgue measure.

Problem 4. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions such that, for each $n \in \mathbb{N}$ we have $f_n : \mathbb{R} \to \mathbb{R}$ and $f_n \in L^2(\mathbb{R}, dx)$. Assume that for every $m, n \in \mathbb{N}$ such that $m \ge n$ one has that

$$\|f_n - f_m\|_2 < 2^{-n},$$

where generically, $\|g\|_2$ denotes the L^2-norm of the function $g \in L^2(\mathbb{R}, dx)$.

(a) Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges point-wise a.e. on $\mathbb{R}$.

(b) Let $\{u_n\}_{n\in\mathbb{N}}$ be an orthonormal set of functions, also in $L^2(\mathbb{R}, dx)$. Does the series

$$\sum_{n=1}^{\infty} \frac{u_n(x)}{2^n}, \qquad x \in \mathbb{R}, \tag{1}$$

converge point-wise a.e. on $\mathbb{R}$?

(c) If $x \in \mathbb{R}$ is a point where the series from (1) converges we denote by $F(x)$ its value. Show that $F \in L^2(\mathbb{R}, dx)$ and compute its norm $\|F\|_2$ explicitly.

Problem 5. (a) Compute the integral

$$I(r) = \int_0^\infty \frac{dx}{1+x^r}, \quad \text{where} \quad r > 1$$

Hint: Use contour integral techniques. More specifically, consider a contour integral over $[0,\infty) \cup \{z \in \mathbb{C} : z = te^{2\pi i/r}, t \in [0,\infty)\}$.

(b) Compute

$$\lim_{r\to\infty} I(r)$$

either directly by examining the limit of the integral above, or by using your answer to part (a).

Problem 6. Let $D_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be the right half plane and assume that $f : \overline{D_+} \to \mathbb{C}$ is analytic in D_+, continuous on the closure $\overline{D}_+$ of D_+, and satisfies

$$|f(z)| \le \begin{cases} 1 & \text{if} \quad \operatorname{Re} z = 0, \\ \ln|z| & \text{if} \quad |z| \ge 3. \end{cases}$$

Show that in fact

$$|f(z)| \le 1 \qquad \text{in} \quad D_+.$$

Hint: Consider the function $g_\varepsilon : \overline{D_+} \to \mathbb{C}$ given by $g_\varepsilon(z) = \frac{f(z)}{(1+z)^\varepsilon}$ for $\varepsilon > 0$.

Problem 7. Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence of analytic functions in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$.

(a) Let $0 < r < 1$. Show that there exist a subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ of the above sequence which converges

$$\lim_{j\to\infty} f_{n_j}(z) = g_r(z), \quad z \in D_r$$

where $D_r := \{z \in \mathbb{C} : |z| < r\}$ and $g_r : D_r \to \mathbb{C}$ is an analytic function.

(b) Does there exist a subsequence of the given sequence $\{f_n\}$ which converges point-wise to a function $g(z)$ analytic on D?