1. Suppose that $(X, \mathcal{A}, \mu)$ is a measure space, with $\mathcal{A}$ a σ-algebra, μ a finite measure. Let $\mathcal{A}_0 \subset \mathcal{A}$ be a sub-algebra of sets which generates $\mathcal{A}$; thus $\mathcal{A}$ is the smallest σ-algebra generated by $\mathcal{A}_0$.

   Let $\mathcal{B} \subset \mathcal{A}$ be the collection of subsets of X with the property that for every ϵ and $B \in \mathcal{B}$, there is an $A \in \mathcal{A}_0$, with
$$\mu(A\Delta B) < \epsilon.$$
   (Here, Δ is the symmetric difference, $C\Delta D = (C - D) \cup (D - C)$.)

   Show that $\mathcal{B}$ is a σ-algebra containing $\mathcal{A}_0$, in particular that it is an algebra, and that it is closed under countable (increasing) unions.

   Hint: You may assume that (do not prove!):
$$\begin{aligned}
\mu(C\Delta D) &= \mu(C^c\Delta D^c)\\
\mu\left((D_1 \cup D_2)\Delta(E_1 \cup E_2)\right) &\leq \mu(D_1\Delta E_1) + \mu(D_2\Delta E_2)\\
|\mu(C\Delta E) - \mu(D\Delta E)| &\leq \mu(C\Delta D)
\end{aligned}$$

2. (a) Let $f(x)$ be a real-valued differentiable function on a neighborhood of $[a, b]$, and assume that $\frac{df}{dx}(a) < 0$, and $\frac{df}{dx}(b) > 0$. Show that there exists a $c \in (a, b)$ such that $\frac{df}{dx}(c) = 0$.

   (b) Suppose again that $\frac{df}{dx}(a) < \frac{df}{dx}(b)$ and that λ satisfies $\frac{df}{dx}(a) < \lambda < \frac{df}{dx}(b)$. Show that there exists a $c \in (a, b)$ with $\frac{df}{dx}(c) = \lambda$.

3. Let (X, μ) be a measure space.

   (a) Show that if g is a non-negative square-integrable function on X, then
$$\int_{\{x:g(x)\geq\lambda\}} g(x)d\mu(x) \leq \frac{1}{\lambda}\int_X g^2(x)d\mu(x).$$
   Suppose now that (X, μ) is a *finite* measure space and let $\{f_n\}_{n=1,2,...}$ be a sequence of *non-negative* functions which are both integrable and square-integrable, i.e., in $L^1(X) \cap L^2(X)$, that they converge pointwise to a function $f(x)$, and that the limits
$$\lim_{n\to\infty}\int f_n(x)d\mu(x) = L \text{ and } \lim_{n\to\infty}\int f_n^2(x)d\mu(x) = M$$
   exist.

(b) Show that

$$\lim_{n\to\infty} \int f_n(x)d\mu(x) = \int f(x)d\mu(x).$$

Hints: Why is

$$\int f(x)d\mu(x) \leq \liminf \int f_n(x)d\mu(x)?$$

To show an inequality in the other direction: Define cut-off functions:

$$f_{n,\lambda}(x) \equiv \begin{cases} f_n(x), & f_n(x) \leq \lambda, \\ \lambda \text{ otherwise,} \end{cases}$$

which converge pointwise to

$$f_\lambda(x) \equiv \begin{cases} f(x), & f(x) \leq \lambda, \\ \lambda \text{ otherwise,} \end{cases}$$

Show that, given ϵ, there is a λ such that (again, μ is a finite measure and use part (a))

$$\begin{aligned} \limsup_n \int f_n(x)d\mu(x) &\leq \limsup_n \int f_{n,\lambda}(x)d\mu(x) + \epsilon \\ &\leq \int f_\lambda(x)d\mu(x) + \epsilon, \end{aligned}$$

which is clearly

$$\leq \int f(x)d\mu(x) + \epsilon.$$

4. Let f be an entire function on the complex plane, and suppose there is a positive integer N such that

$$\frac{f(z)}{z^N} \to 0 \text{ as } |z| \to \infty.$$

Find an upper bound on the number of zeros of f (counting multiplicity) in the complex plane in terms of N. Is the bound sharp?

5. Let $\mathcal{H}$ be the Hilbert space of $L^2(D, \mu)$ functions on the unit disc $D = \{z \in \mathcal{C} : |z| < 1\}$, with μ two-dimensional measure, $d\mu = dx\, dy = r\, dr\, d\theta$ in polar coordinates, with $\|f\|_{L^2(D)}$ its L^2-norm. Suppose that $f \in \mathcal{H}$ is moreover analytic in D.

   (a) Show that, for $0 < r < 1$

   $$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta,$$

   and that for $0 < r_1 < 1$,

   $$f(0) = \frac{1}{\pi r_1^2} \int_0^{2\pi} \int_0^{r_1} f(re^{i\theta}) r\, dr\, d\theta.$$

   (b) Show that

   $$|f(0)| \leq \frac{1}{\sqrt{\pi} r_1} \|f\|_{L^2(D)},$$

   for all $f \in \mathcal{H}$ which are analytic in D.

6. Use the method of residues to evaluate

   $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4}\, dx.$$

   Show all estimates.

7. Suppose that f is analytic on $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 3\}$, and assume that $|f(z)| \leq 1$ for $|z| = 1$ and $|f(z)| \leq 9$ for $|z| = 3$. Prove that $|f(2i)| \leq 4$.