1. (10 points) Classify, up to isomorphism, all finite groups of order 2p, where p is a prime number.

2. (12 points, 6 points each) Consider the ring
\[ R = \mathbb{Z}[\sqrt{-11}] = \{ m + n\sqrt{-11} \mid m, n \in \mathbb{Z} \}. \]
(a) Is R a UFD? Give arguments for your answer.
(b) Exhibit an ideal I in R which is not principal. Show that your I is not principal.

3. (15 points, 5 points each) Decide in each of the following three cases whether the given polynomial is irreducible. Include arguments.
(a) \( x^2 - 2i \) in \( \mathbb{Z}[i][x] \);
(b) \( x^3 - 49x^2 + (3 + \sqrt{2})x + 7 \) in \( \mathbb{Z}[\sqrt{2}][x] \);
(c) \( x^2 + xy + y^2 \) in \( \mathbb{C}[x, y] \).

4. (12 points) Let A be a finite abelian group of order n, p a prime divisor of n and \( n = p^k m \) with \( k, m \in \mathbb{N} \) such that \( (p, m) = 1 \). Denote by \( A_p \) the Sylow p-subgroup of A.
(a) (8 points) Show that the abelian groups \( A_p \) and \( \mathbb{Z}/p^k\mathbb{Z} \otimes_\mathbb{Z} A \) are isomorphic.
(b) (4 points) Describe \( \mathbb{Z}/p\mathbb{Z} \otimes_\mathbb{Z} A \) as an abelian group without using tensor products but (certain) invariants of A.
5. (16 points) We set $M := M_3(\mathbb{Q})$ and denote by $0$ the zero matrix and by $I$ the identity matrix of $M$.

(a) (2 points) Prove or disprove: If $A \in M$ satisfies $A^6 = 0$, then also $A^3 = 0$.

(b) (4 points) Classify, up to similarity, all matrices in $M$ satisfying $A^6 = 0$. Exhibit one representative for each such similarity class.

(c) (2 points) Prove or disprove: If $A \in M$ satisfies $A^6 = I$, then also $A^3 = I$.

(d) (8 points) Classify, up to similarity, all matrices in $M$ satisfying $A^6 = I$. Exhibit one representative for each such similarity class.

6. (10 points) Let $n \geq 2$ be a natural number, $F$ a field and $A = (a_{ij}) \in M_n(F)$ the matrix with entries $a_{ij} = j \cdot 1_F \in F$ for all $1 \leq i, j \leq n$.

Determine the characteristic polynomial, the minimal polynomial and the JCF of $A$.

Hint: The result may depend on the characteristic of $F$.

7. (15 points)

(a) (8 points) Construct, using cyclotomic fields, a Galois extension $K$ of $\mathbb{Q}$ of degree 3. Include arguments.

(b) (7 points) Find, explicitly, a polynomial $f(x) \in \mathbb{Q}[x]$ such that your field $K$ in (a) is the splitting field of $f(x)$ over $\mathbb{Q}$.

8. (10 points) Let $M \mid K \mid F$ be a tower of finite field extensions such that $M \mid F$ and $K \mid F$ are both Galois. Assume that the Galois group $G(M|K)$ is cyclic. Prove that $L \mid F$ is Galois for every intermediate field $L$ with $M \mid L \mid K$. 