1. (10 points) Denote by $D_{2n}$ the dihedral group of order $2n$, $n = 1$ being admitted. For which natural numbers $m$ and $n$ is $D_{4nm}$ isomorphic to the direct product $D_{2m} \times D_{2n}$?

2. (10 points) Let $G$ be a group of order $16 \cdot 11 \cdot 13 \cdot 17$. Assume that $G$ has a normal nonabelian Sylow 2-subgroup. Show that the center of $G$ is nontrivial.

Remark: The claim remains true if $G$ has an abelian normal Sylow 2-subgroup but then it is a bit harder to prove.

3. (16 points) Let $p$ be a prime, $n$ a natural number and $A \in GL_n(\mathbb{F}_p)$ diagonalizable over the algebraic closure $\overline{\mathbb{F}}_p$.

(a) (4 points) Show that the order of $A$ in $GL_n(\mathbb{F}_p)$ is equal to the lcm of the orders of the eigenvalues of $A$ in $\mathbb{F}_p^\times$.

(b) (8 points) Prove that $GL_n(\mathbb{F}_p)$ has an element of order $p^n - 1$ which is diagonalizable over $\overline{\mathbb{F}}_p$.

(c) (4 points) Explicitly construct an element of order 8 of $GL_2(\mathbb{F}_3)$.

4. (12 points) Let $R$ be a commutative ring with 1 which is Artinian, i.e. for any descending chain $I_1 \supseteq I_2 \ldots \supseteq I_n \ldots$ of ideals of $R$ there exists an $n_0$ such that $I_n = I_{n_0}$ for all $n \geq n_0$. Prove the following:

(a) (4 points) If $R$ is an integral domain, then it is a field.

(b) (4 points) Any prime ideal of $R$ is maximal.

(c) (4 points) $R$ has only finitely many maximal ideals.
5. (12 points) Let \( R = \mathbb{Z}[x]/(x^3 + x^2 + 1) \) be the quotient of the polynomial ring \( \mathbb{Z}[x] \) modulo the principal ideal \( (x^3 + x^2 + 1) \).
   (a) (4 points) Is \( R \) an integral domain?
   (b) (8 points) Which of the principal ideals (2), (3), (5) of \( R \) are prime ideals? And which of them are maximal?

6. (15 points) Let \( M|K, M|L, K|F, L|F \) be finite field extensions. Assume that for \( \alpha, \beta \in M, K = F(\alpha), L = F(\beta) \) and \( M = F(\alpha, \beta) \). Set \( a = [K : F] \) and \( b = [L : F] \).
   (a) (3 points) If \( K|F \) and \( L|F \) are Galois, show that also \( M|F \) is Galois.
   (b) (8 points) If \( K|F \) and \( L|F \) are Galois, prove that \( [M : F] \) divides \( ab \).
   (c) (4 points) Give an example of \( M, K, L, F \) as above (but without the Galois assumption) such that \( [M : F] \) does not divide \( ab \).

7. (15 points) Consider the polynomial ring \( R = F[x, y] \) in two variables over the field \( F \) and the ideal \( I = (x, y) \) of \( R \). Let \( \phi : R \rightarrow F \) be the \( F \)-algebra homomorphism with \( \phi(x) = \phi(y) = 0 \), which turns \( F \) into an \( R \)-module.
   (a) (4 points) Show that the \( R \)-modules \( F \otimes_R F \) and \( F \) are isomorphic.
   (b) (2 points) Define maps \( s, t : I \rightarrow F \) by \( s(f) = c_{1,0} \), respectively, \( t(f) = c_{0,1} \) if \( f = \sum_{i,j} c_{i,j} x^i y^j \in I \) (with \( c_{i,j} \in F \) for all \( i, j \in \mathbb{N}_0 \)).
   Verify that \( s \) and \( t \) are \( R \)-module homomorphisms.
   (c) (6 points) Prove that \( x \otimes y - y \otimes x \) is not 0 in \( I \otimes_R I \).
   (d) (3 points) Prove that \( I \) is not a flat \( R \)-module.

8. (10 points) Consider the \( \mathbb{C} \)-vector space \( V = M_2(\mathbb{C}) \) of complex \( 2 \times 2 \) matrices and the linear transformation \( T : V \rightarrow V \) defined by \( T(X) = AX - XA \) for all \( X \in M_2(\mathbb{C}) \), where \( A \) is the matrix \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \).
   Determine (as a \( 4 \times 4 \) matrix) the Jordan canonical form of \( T \).