Algebra general exam  
January 12, 2017

Your name:

- Please show all your work and justify any statements that you make.
- State any theorem you use clearly and fully.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement of an earlier question proven in order to solve a later one.

Sign below the pledge:

"On my honor, I pledge that I have neither given nor received help on this assignment."

1. Consider the polynomial \( f(X) = X^4 - 2X^2 - 6 \). Prove this polynomial is irreducible. Describe the splitting field of this polynomial (including its degree over \( \mathbb{Q} \)), and the Galois group of this splitting field (hint: pay attention to which roots are real and which are complex). (15pt)

2. Consider a field \( K \), and two finite extensions \( L, M \) of \( K \). Consider the \( K \)-algebra \( L \otimes_K M \) (with the usual multiplication \((a \otimes b)(c \otimes d) = ac \otimes bd\)). Prove that \( L \otimes_K M \) is a field if and only if any time an extension \( E/K \) contains subfields \( L' \) and \( M' \) isomorphic to \( L \) and \( M \), the composite \( LM \) has degree \([LM : K] = [L : K][M : K]\). (15pt)

3. Let \( R \) be a commutative ring, and \( M, N \) be \( R \)-modules. Show that for any submodules \( M' \subset M \) and \( N' \subset N \), the induced map \( M \otimes_R N \rightarrow (M/M') \otimes_R (N/N') \) has kernel given by \( M' \otimes N + M \otimes N' \) (hint: use the universal property of tensor products). (10pt)

4. We call a group \( G \) polycyclic if it contains a series of subgroups \( \{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G \) such that \( G_i/G_{i-1} \) is a (possibly infinite) cyclic group.
   (a) Show that a finite group is polycyclic if and only if it is solvable. (5pt)
   (b) Show that \( \mathbb{Q} \) is an example of an abelian group which is not polycyclic. (5pt)

5. Given a finite group \( G \) and two subgroups \( H, K \), the double cosets of \( H \) and \( K \) are the sets of the form \( HgK \) for some \( g \in G \).
   (a) Show that any two double cosets must be equal or disjoint. (5pt)
   (b) Show that the size of any double coset must divide the product of the orders \( \#H \cdot \#K \). (5pt)
   (c) Find an example of a double coset whose size does not divide the order \( \#G \). (5pt)

6. (a) Find the smallest integer \( n \) such that \( S_n \) has a subgroup of order 10, but \( S_k \) for \( k < n \) does not. (5pt)
   (b) Find the smallest integer \( m \) such that \( S_m \) has an element of order 10, but \( S_k \) for \( k < m \) does not. (5pt)
7. Consider the matrix
\[
A = \begin{bmatrix}
12 & 4 & -16 \\
4 & 3 & -7 \\
8 & 3 & -11j
\end{bmatrix}.
\]
(a) Find the characteristic and minimal polynomials of this polynomial and its Jordan normal form. (8pt)
(b) Consider map \( \mathbb{Z}^3 \to \mathbb{Z}^3 \) induced by \( A \). Describe the kernel and cokernel of this map as a sum of copies of \( \mathbb{Z} \) and \( \mathbb{Z}/n\mathbb{Z} \). (7pt)

8. Let \( A \) be the ring of \( n \times n \) matrices over a field \( F \).
(a) Show the right ideals of \( A \) are precisely the subsets of the form
\[
\{ X \in A \mid \text{image}(X) \subset V \}
\]
where \( V \) ranges over all linear subspaces of \( F^n \). (5 pts)
(b) Show the left ideals of \( A \) are precisely the subsets of the form
\[
\{ X \in A \mid \text{kernel}(X) \supset W \}
\]
where \( W \) ranges over all linear subspaces of \( F^n \). (5 pts)
(c) Show that \( A \) is a simple ring: its only 2-sided ideals are \( A \) itself, and \( \{0\} \). (5 pts)