1. (10 pts) Find all maximal ideals of \( \mathbb{Z}[i] \) which contain 182. Find minimal generators for these ideals.

2. (10 pts) Let \( r_1, r_2, r_3 \) be the roots of the cubic polynomial \( X^3 + 10X^2 - 5X + 4 \). Find the cubic polynomial with rational coefficients whose roots are \( r_1^2, r_2^2, r_3^2 \).

3. (20 pts)
   a) Let \( G \) be a group of order \( 2n \), where \( n \) is odd and \( n > 1 \). Prove that \( G \) cannot be simple. (Hint: consider elements of order 2 in the regular representation of \( G \) in \( S_{2n} \).)
   b) Let \( G = \mathbb{Z}_n^* \) denote the group of units in \( \mathbb{Z}_n \). Find all integers \( n \) such that \( x^2 = 1 \) for all \( x \in G \).

4 (15 pts). Find the characteristic polynomial, the minimal polynomial, and the Jordan canonical form of the matrix (over the complex numbers)

\[
A = \begin{pmatrix}
1 & 2 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 0 & 2 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}.
\]

5 (10 pts). Let \( R \) be a commutative ring with identity, and let \( I \) be a nilpotent ideal, i.e., \( I^k = 0 \) for some \( k \). Let \( M, N \) be two \( R \)-modules, and let \( f : M \to N \) be an \( R \)-homomorphism. Suppose that the induced homomorphism from \( M/IM \) to \( N/IN \) is surjective. Prove that \( f \) is surjective.
6. (10 pts) Let $F$ be a field and $A$ an $n$ by $n$ matrix with coefficients in $F$. Assume that $A$ has only one invariant factor. Prove that for every $n$ by $n$ matrix $B$ with coefficients in $F$ such that $AB = BA$ there is a polynomial $p(t) \in F[t]$ such that $p(A) = B$. (Hint: consider the structure of $V = F^n$ as a $k[A]$-module. Use that an endomorphism is determined by its action on a basis.)

7 (10 pts). Let $V$ be a finite dimensional vector space over a field $F$ and let $V^*$ be its dual. For $v \in V$ and $f \in V^*$, denote by $\phi_{v,f}$ the endomorphism of $V$ defined by $\phi_{v,f}(w) = f(w)v$ for $w \in V$. Prove that there exists a well-defined $F$-linear map $\Phi : V \otimes_F V^* \rightarrow \text{End}_F(V)$ satisfying $\Phi(v \otimes f) = \phi_{v,f}$ for all $v \in V$ and $f \in V^*$. Prove that $\Phi$ is an isomorphism.

8 (15 pts). Consider the polynomial $x^6 - 3$ over the rational numbers. What is the degree of its splitting field, and what is the splitting field? Describe the Galois group of the splitting field as a subgroup of the symmetric group $S_6$. Is it abelian?