ALGEBRA GENERAL EXAM, JANUARY 6, 2013, 9AM–1PM

Directions.

- Show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will hurt your grade.
- You may assume the statement in an earlier part proven in order to do a later part.
- Do each problem on a separate one-sided sheet of paper, and staple them together in the correct order.

Problem 1 (10 points)**.** *Let K be an algebraically closed field. Show that any element of finite order in $GL_n(K)$ is diagonalizable. (Hint: Jordan Form!).*

Problem 2 (10 points)**.** *Find all ring homomorphisms*

(1) *from $\mathbb{Z}$ to $\mathbb{Z}/30\mathbb{Z}$;*
(2) *from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}$.*

Problem 3 (10 points)**.** *Let $\mathbb{Q}(\sqrt{-2})$ be a quadratic field with associated ring of integer $\mathcal{O} = \mathbb{Z}[\sqrt{-2}]$. Prove that $\mathcal{O}$ is a Euclidean Domain . (Hint: use the field norm.)*

Problem 4 (10 points)**.** *Prove that if R is a principle ideal domain (P.I.D.) and D is a multiplicatively closed subset of R with $0 \notin D$, then $D^{-1}R$ is also a P.I.D.*

Problem 5 (10 points)**.**

(1) *Consider the abelian group*

$$A = \prod_{n \geq 2} \mathbb{Z}/n\mathbb{Z}.$$

Show that this is not a torsion group by exhibiting an element of infinite order.

(2) *Show that $\mathbb{Q} \otimes_{\mathbb{Z}} A \neq 0$. (Hint: this is $S^{-1}A$ for $S = \mathbb{Z} - \{0\}$.) Bonus: Determine $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} A)$.*

Problem 6 (10 points)**.** *Let $K|F$ be an extension of finite fields, and let L, M be subfields of K containing F. Assume that $L \cap M = F$.*

(1) *Show that the degrees $[L : F]$ and $[M : F]$ are relatively prime.*
Hint: How many subfields of a given order does a finite field have?
(2) *Now assume additionally that $L = F(\alpha)$, $M = F(\beta)$ and $K = F(\alpha, \beta)$ for some $\alpha, \beta \in K$. Prove that $K = F(\alpha + \beta)$.*

Problem 7 (10 points)**.** *Consider the real number $u = \sqrt{3 + \sqrt{11}}$.*

(1) *Determine the minimal polynomial for u over $\mathbb{Q}$, and justify that it is the minimal polynomial.*
(2) *Is $\mathbb{Q}[u]$ the splitting field for the minimal polynomial of u? (Hint: consider which roots are real and which are complex.)*
(3) *Determine the Galois group of the splitting field. (Hint: What is the degree of the field extension?)*

Problem 8 (10 points)**.** *Use the semidirect product constructions to classify the groups of order 44. (Hint: start by analyzing Sylow subgroup structures.)*