Algebra General Exam
August 19, 2013

Directions.

• Please show all your work and justify any statements that you make
• State clearly and fully any theorem you use
• Vague statements and hand-waving arguments will not be viewed favorable
• You may assume the statement for any early part of a problem in order to do a later part

Do each problem on a separate sheet of paper

(1) Let $p$ be an odd prime and $G$ a nonabelian group of order $p^3$.
(a) (4 points) Prove that $|Z(G)| = p$
(b) (4 points) Prove that $Z(G) = [G, G]$.
(2) (5 points) Let $K$ and $L$ be fields of characteristic 0. Prove that $K \otimes L$ is nonzero.
(3) If $G$ is a group, then there is a natural action of $\Sigma_n$ on $G^n$ given by permuting the factors. Define the wreath product $G \wr \Sigma_n$ to be

$$G \wr \Sigma_n = G^n \rtimes \Sigma_n$$

using this action of $\Sigma_n$ on $G^n$.
(a) (3 points) If $X$ is a $G$-set, show that $X^n$ is naturally a $G \wr \Sigma_n$ set by combining two actions: $G^n$ on $X^n$ via

$$(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_n) = (g_1 x_1, \ldots, g_n x_n)$$

for $(g_1, \ldots, g_n) \in G^n$ and $(x_1, \ldots, x_n) \in X^n$, and

$\Sigma_n$ on $X^n$ via

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

where $\sigma \in \Sigma_n$.
(b) (3 points) Show that $\Sigma_n \wr \Sigma_m$ embeds into $\Sigma_{nm}$.
(c) (3 points) Identify $\Sigma_2 \wr \Sigma_2$ with a more familiar group
(d) (2 points) Determine the order of $G \wr \Sigma_n$ as a function of the orders of $G$ and $n$
(e) (2 points) Bonus: Determine (no proof needed) the $p$-Sylow subgroup of $\Sigma_{p^{k+1}}$ as a function of $k$. Provide no more than a sentence of justification.

(4) Let $K$ be a field, and let $M_n(K)$ be the ring of $n \times n$ matrices with entries in $K$. For this problem, let $D \in M_n(K)$ be diagonalizable (over $K$) and, for each eigenvalue $\lambda$ of $D$, let

$$E_\lambda := \{ v \in K^n \mid Dv = \lambda v \}$$

be the corresponding eigenspace.
(a) (4 points) For any $A \in M_n(K)$, show that $AD = DA$ if and only if $A(E_\lambda) \subseteq E_\lambda$ for all eigenvalues $\lambda$ of $D$.
(Hint: For the “if” part, you may use that $AD = DA$ if $ADv = DAv$ for all $v \in K^n$.)

(b) (4 points) If $A$ is also diagonalizable and $AD = DA$, show that $A$ and $D$ are simultaneously diagonalizable (that is, there is a matrix $P$ such that both $PAP^{-1}$ and $PDP^{-1}$ are diagonal). Provide a counter-example showing that this need not be the case if the matrices do not commute.

(c) (3 points) If $D$ is invertible, show that the centralizer of $D$ in $GL_n(K)$ is isomorphic to a direct product $GL_{n_1}(K) \times \ldots \times GL_{n_r}(K)$, where $n_1 + \ldots + n_r = n$. Also show that each of these products can be realized as the centralizer of some (appropriately chosen) $D$, provided that $K$ has at least $n + 1$ elements.

(5) Let $F$ be a field and $f(x) = x^4 + 1 \in F[x]$.
(a) (3 points) Determine for which characteristic of $F f(x)$ is separable.
(b) (4 points) Assume that $f(x)$ is separable and irreducible over $F$, and denote by $K$ the splitting field of $f(x)$ over $F$. Determine the Galois group $Gal(K|F)$.
(c) (4 points) If $f(x)$ is irreducible over $F$, prove first that $F$ is infinite, and then that the characteristic of $F$ is 0.

(6) Let $p$ be a prime and $\zeta$ a primitive $p^{th}$ root of unity (in $\mathbb{C}$). Set $R := \mathbb{Z}[\zeta]$ and $K := \mathbb{Q}(\zeta)$.
(a) (2 points) Show that $R$ is a free $\mathbb{Z}$-module and $R \cap \mathbb{Q} = \mathbb{Z}$.
(b) (2 points) Identify $Gal(K|\mathbb{Q})$ and show that the natural action of $Gal(K|\mathbb{Q})$ on $K$ sends elements of $R$ to itself (hence giving an action of $Gal(K|\mathbb{Q})$ on $R$).
(c) (3 points) For any two integers $m, n$ which are not divisible by $p$, show that the quotient $(1 - \zeta^m)/(1 - \zeta^n)$ is an element of $R$.
Hint: Reduce to the case where $n$ divides $m$.
(d) (2 points) Verify that $p = (1 - \zeta) \ldots (1 - \zeta^{p-1})$.
Hint: manipulate the cyclotomic polynomial associated to $\zeta$.
(e) (3 points) Prove that $1 - \zeta$ is not a unit of $R$.
(f) (2 points) Prove (using norms) that $1 - \zeta$ is an irreducible element of $R$. (It is true, but harder to prove, that $1 - \zeta$ is in fact a prime element of $R$.)