Algebra general exam. January 9, 2013, 9am -1pm

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let p be a prime and let S_{2p} denote the symmetric group on $2p$ elements.

(a) (2 pts) Find the order of a p-Sylow subgroup of S_{2p}.

(b) (5 pts) Describe explicitly a p-Sylow subgroup of S_{2p} (providing a generating set counts as explicit description, but make sure to prove that your subgroup is indeed p-Sylow).

(c) (2 pts) Consider the set of elements of order p in S_{2p} – clearly, it is a union of conjugacy classes. How many conjugacy classes does it consist of?

(d) (5 pts) Now consider the set of elements of order p in the alternating group A_{2p}. How many conjugacy classes (of A_{2p}) does it consist of? Make sure to justify your answer.

Hint: Distinguish between the cases $p = 2$ and $p > 2$.

2. In both parts of this problem R is a commutative domain with 1 and K is the field of fractions of R.

(a) (5 pts) Let $R = \mathbb{Z}[t]$, the ring of polynomials over $\mathbb{Z}$ in one variable. Let $p(x) = x^n + r_{n-1}x^{n-1} + \ldots + r_0 \in R[x]$ be a monic polynomial with coefficients in R, and suppose that $p(\alpha) = 0$ for some $\alpha \in K$. Prove that $\alpha \in R$.

(b) (4 pts) Now let $R = \mathbb{Z}[\sqrt{-3}]$. Find a monic polynomial $p(x) \in R[x]$ which has a root in K, but has no root in R (and prove that $p(x)$ has required properties). **Hint:** There actually exists a quadratic polynomial with integer coefficients with required property.

3. (6 pts) Let F be a field, d a positive integer, and $f_1, f_2, \ldots \in F[x_1, \ldots, x_d]$ an infinite sequence of polynomials in $F[x_1, \ldots, x_d]$. Given a positive integer n, let S_n be the set of all d-tuples $(a_1, \ldots, a_d) \in F^d$ satisfying the following system of equations:

$$f_i(a_1, \ldots, a_d) = 0 \text{ for each } 1 \le i \le n-1 \text{ and } f_n(a_1, \ldots, a_d) = 1.$$

Prove that there exists an integer N such that the set S_n is empty for all $n \ge N$. **Hint:** Noetherian rings.

4. Let p be a prime, $\mathbb{F}_p$ a finite field of order p, and let F be a fixed algebraic closure of $\mathbb{F}_p$. For $n \in \mathbb{N}$, denote by $\mathbb{F}_{p^n}$ the unique subfield of order p^n inside F.

(a) (3 pts) Prove that $\mathbb{F}_{p^n} \cup \mathbb{F}_{p^m}$ is a subfield if and only if m divides n or n divides m.

(b) (4 pts) For a subset S of $\mathbb{N}$, let

$$F(S) = \bigcup_{n \in S} \mathbb{F}_{p^n}.$$

Give an example (with proof) of an infinite set S for which $F(S)$ is a subfield and $F(S) \neq F$.

5. Let $\omega = e^{2\pi i/3}$ and consider the field $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$.

(a) (2 pts) Prove that $[K : \mathbb{Q}] = 6$.

(b) (2 pts) Prove that $K/\mathbb{Q}$ is a Galois extension.

(c) (3 pts) Let M/L be any finite Galois extension. Prove that an element $\gamma \in M$ is primitive for M/L (that is, $L(\gamma) = M$) if and only if $\sigma(\gamma) \neq \gamma$ for any $\sigma \in \mathrm{Gal}(M/L) \setminus \{1\}$.

(d) (4 pts) Now prove that $\gamma = \sqrt[3]{2} + \omega$ is a primitive element for $K/\mathbb{Q}$.

(e) (3 pts) Let $x^6 + a_5x^5 + \ldots + a_0$ be the minimal polynomial of γ over $\mathbb{Q}$. Prove that $a_5 = 3$ without actually computing the minimal polynomial.

6. Let F be an algebraically closed field and $A \in Mat_n(F)$ an $n \times n$ matrix over F for some $n \geq 2$.

(a) (6 pts) Prove that there exist a diagonalizable matrix D and a nilpotent matrix N (that is, $N^k = 0$ for some $k \in \mathbb{N}$) such that $A = D + N$ and D and N commute, that is, $DN = ND$.

(b) (4 pts) Assume that A itself is diagonalizable. Prove that if D and N satisfy the conditions of part (a), then $N = 0$ (and hence $D = A$). **Hint:** You may use the following fact without proof: if two diagonalizable matrices X and Y commute, then they are simultaneously diagonalizable, that is, there exists an invertible matrix Q such that $Q^{-1}XQ$ and $Q^{-1}YQ$ are both diagonal.