Algebra general exam. January 9, 2013, 9am -1pm

Directions.

• Please show all your work and justify any statements that you make.
• State clearly and fully any theorem you use.
• Vague statements and hand-waving arguments will not be appreciated.
• You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let \( p \) be a prime and let \( S_{2p} \) denote the symmetric group on \( 2p \) elements.
   (a) (2 pts) Find the order of a \( p \)-Sylow subgroup of \( S_{2p} \).
   (b) (5 pts) Describe explicitly a \( p \)-Sylow subgroup of \( S_{2p} \) (providing a generating set counts as explicit description, but make sure to prove that your subgroup is indeed \( p \)-Sylow).
   (c) (2 pts) Consider the set of elements of order \( p \) in \( S_{2p} \) – clearly, it is a union of conjugacy classes. How many conjugacy classes does it consist of?
   (d) (5 pts) Now consider the set of elements of order \( p \) in the alternating group \( A_{2p} \). How many conjugacy classes (of \( A_{2p} \)) does it consist of? Make sure to justify your answer.
   \textbf{Hint:} Distinguish between the cases \( p = 2 \) and \( p > 2 \).

2. In both parts of this problem \( R \) is a commutative domain with 1 and \( K \) is the field of fractions of \( R \).
   (a) (5 pts) Let \( R = \mathbb{Z}[t] \), the ring of polynomials over \( \mathbb{Z} \) in one variable. Let \( p(x) = x^n + r_{n-1}x^{n-1} + \ldots + r_0 \in R[x] \) be a monic polynomial with coefficients in \( R \), and suppose that \( p(\alpha) = 0 \) for some \( \alpha \in K \). Prove that \( \alpha \in R \).
   (b) (4 pts) Now let \( R = \mathbb{Z}[\sqrt{-3}] \). Find a monic polynomial \( p(x) \in R[x] \) which has a root in \( K \), but has no root in \( R \) (and prove that \( p(x) \) has required properties). \textbf{Hint:} There actually exists a quadratic polynomial with integer coefficients with required property.

3. (6 pts) Let \( F \) be a field, \( d \) a positive integer, and \( f_1, f_2, \ldots \in F[x_1, \ldots, x_d] \) an infinite sequence of polynomials in \( F[x_1, \ldots, x_d] \). Given a positive integer \( n \), let \( S_n \) be the set of all \( d \)-tuples \( (a_1, \ldots, a_d) \in F^d \) satisfying the following system of equations:
   \[ f_i(a_1, \ldots, a_d) = 0 \text{ for each } 1 \leq i \leq n - 1 \text{ and } f_n(a_1, \ldots, a_d) = 1. \]
   Prove that there exists an integer \( N \) such that the set \( S_n \) is empty for all \( n \geq N \). \textbf{Hint:} Noetherian rings.
4. Let \( p \) be a prime, \( \mathbb{F}_p \) a finite field of order \( p \), and let \( F \) be a fixed algebraic closure of \( \mathbb{F}_p \). For \( n \in \mathbb{N} \), denote by \( \mathbb{F}_{p^n} \) the unique subfield of order \( p^n \) inside \( F \).

(a) (3 pts) Prove that \( \mathbb{F}_{p^n} \cup \mathbb{F}_{p^m} \) is a subfield if and only if \( m \) divides \( n \) or \( n \) divides \( m \).

(b) (4 pts) For a subset \( S \) of \( \mathbb{N} \), let

\[
F(S) = \bigcup_{n \in S} \mathbb{F}_{p^n}.
\]

Give an example (with proof) of an infinite set \( S \) for which \( F(S) \) is a subfield and \( F(S) \neq F \).

5. Let \( \omega = e^{2\pi i/3} \) and consider the field \( K = \mathbb{Q}(\sqrt[3]{2}, \omega) \).

(a) (2 pts) Prove that \([K : \mathbb{Q}] = 6\).

(b) (2 pts) Prove that \( K/\mathbb{Q} \) is a Galois extension.

(c) (3 pts) Let \( M/L \) be any finite Galois extension. Prove that an element \( \gamma \in M \) is primitive for \( M/L \) (that is, \( L(\gamma) = M \)) if and only if \( \sigma(\gamma) \neq \gamma \) for any \( \sigma \in \text{Gal}(M/L) \setminus \{1\} \).

(d) (4 pts) Now prove that \( \gamma = \sqrt[3]{2} + \omega \) is a primitive element for \( K/\mathbb{Q} \).

(e) (3 pts) Let \( x^5 + a_5 x^5 + \ldots + a_0 \) be the minimal polynomial of \( \gamma \) over \( \mathbb{Q} \). Prove that \( a_5 = 3 \) without actually computing the minimal polynomial.

6. Let \( F \) be an algebraically closed field and \( A \in \text{Mat}_n(F) \) an \( n \times n \) matrix over \( F \) for some \( n \geq 2 \).

(a) (6 pts) Prove that there exist a diagonalizable matrix \( D \) and a nilpotent matrix \( N \) (that is, \( N^k = 0 \) for some \( k \in \mathbb{N} \)) such that \( A = D + N \) and \( D \) and \( N \) commute, that is, \( DN = ND \).

(b) (4 pts) Assume that \( A \) itself is diagonalizable. Prove that if \( D \) and \( N \) satisfy the conditions of part (a), then \( N = 0 \) (and hence \( D = A \)).

**Hint:** You may use the following fact without proof: if two diagonalizable matrices \( X \) and \( Y \) commute, then they are simultaneously diagonalizable, that is, there exists an invertible matrix \( Q \) such that \( Q^{-1}XQ \) and \( Q^{-1}YQ \) are both diagonal.