Algebra general exam. August 22nd 2012, 9am-1pm

Directions.

• Please show all your work and justify any statements that you make.
• State clearly and fully any theorem you use.
• Vague statements and hand-waving arguments will not be appreciated.
• You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. For a positive integer $n$, denote by $S_n$ the symmetric group on \{1, 2, \ldots, n\}.
   Let $p > 2$ be a prime number.
   (a) (2 pts) Give an example of a non-cyclic group of order $2p$.
   (b) (5 pts) Find the smallest $n$ for which $S_n$ contains a cyclic subgroup of order $2p$.
   (c) (7 pts) Find the smallest $n$ for which $S_n$ contains some subgroup of order $2p$.

   In both (b) and (c), if $n$ is your answer, explain clearly why $S_n$ contains a desired subgroup and why $S_m$ for $m < n$ does not contain such subgroup.

2. (8 pts) Let $G$ be a finite group and $p$ a prime divisor of $|G|$. Assume that every element of $G$ of $p$-power order is contained in a normal $p$-subgroup of $G$. Show that $G$ has only one Sylow $p$-subgroup.

3. Let $k$ be a field and $R = k[x, y]/(x^5 - y^2)$.
   (a) (5 pts) Prove that $R$ is isomorphic to the subring $k[t^2, t^5]$ of $k[t]$ (the polynomials in one variable over $k$).
   (b) (8 pts) Prove that $R$ is not isomorphic to $k[t]$ (as a ring).

4. (8 pts) Let $R$ be a commutative ring with 1. Let $N$ be the nilradical of $R$, that is, $N$ is the set of all nilpotent elements of $R$ (including 0). You may use without proof that $N$ is the intersection of all prime ideals of $R$. Prove that the following conditions are equivalent:
   (i) $R$ has just one prime ideal.
   (ii) $R/N$ is a field.

5. Recall that if $R$ is a commutative ring with 1 and $A$ and $B$ are $R$-algebras, then $A \otimes_R B$ also has the natural structure of an $R$-algebra.
   (a) (3 pts) Let $K$ and $L$ be fields of different characteristics. Prove that $K \otimes_Z L = \{0\}$.
   (b) (6 pts) Let $K$ and $L$ be fields of the same positive characteristic $p$. Prove that $K \otimes_Z L$ can be provided in a natural way with the structure of an $\mathbb{F}_p$-algebra, and that this $\mathbb{F}_p$-algebra is isomorphic to $K \otimes_{\mathbb{F}_p} L$. Deduce that $K \otimes_Z L$ is nonzero.
(c) (5 pts) Find an example of commutative rings $A$ and $B$ which are NOT fields such that $A \otimes \mathbb{Z} B$ is a field. **Hint:** Use a suitable property of tensor products involving direct sums.

6. Let $p$ be an odd prime number and $n$ an integer $\geq 2$.
   (a) (5 pts) Show that $GL_n(\mathbb{Q})$ has an element of order $p$ if and only if $n \geq p - 1$.
   (b) (5 pts) Show that there is an $A \in GL_n(\mathbb{Q})$ of order $p$ which does NOT have 1 as an eigenvalue if and only if $p - 1$ divides $n$.
   (c) (5 pts) Let $A \in GL_4(\mathbb{Q})$ be an element of order 5. Prove that the complex JCF of $A$ is independent of such $A$ (up to permutation of blocks) and write it down.

7. Let $K/F$ be a field extension, let $\alpha, \beta \in K \setminus F$ be algebraic over $F$, and let $p = \deg_F(\alpha)$ and $q = \deg_F(\beta)$. Suppose that $p$ and $q$ are distinct primes and that $p > q$.
   (a) (2 pts) Prove that $[F(\alpha, \beta) : F] = pq$.
   (b) (6 pts) Prove that $\deg_F(\alpha\beta) = p$ or $pq$.
   (c) (4 pts) Give an example showing that it MAY happen that $\deg_F(\alpha\beta) = p$.

8. In this problem you may use the following fact without proof: for any group $G$ there exists a Galois extension $M/L$ with $Gal(M/L) \cong G$.
   (a) (8 pts) Prove that there exists a field extension $K/F$ such that $[K : F] = 4$ and there are no intermediate fields between $F$ and $K$ other than $F$ and $K$. **Hint:** First reduce the question to a purely group-theoretic problem. Partial credit will be given for such reduction.
   (b) (3 pts) Is it possible to construct an extension satisfying (a) if $F$ is finite? Justify your answer.
   (c) (5 pts) Is it possible to construct an extension satisfying (a) if $K$ is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$ for some $n$ (where $\zeta_n$ is a primitive $n^{th}$ root of unity)? Justify your answer.