General Exam: Algebra
August 15, 2011

Instructions. You have 4 hours for the exam. Each part of each question is worth 4 points. Do each question on a separate sheet of paper (one side only please), and staple the sheets together in the correct order.

On this exam, all rings R have an identity 1 by fiat.

1. Let M be a simple (left) module for a ring R. This means that M has no submodules apart from 0 and M.

(a) Prove that $M \cong R/I$ where I is a maximal left ideal of R..

(b) Show that $E := \mathrm{End}_R(M)$ is a division ring, i. e., every nonzero element of E is invertible.

2. The following question concerns symmetric groups. You can assume as given the fact that any permutation in S_n can be written (uniquely up to order) as a (commuting) product of disjoint cycles (of varying lengths). Otherwise, your argument should be self-contained.

(a) For $n \geq 2$, show that the symmetric group S_n is generated by the transpositions (i, j), $1 \leq i < j \leq n$.

(b) For $n \geq 3$, show that the alternating group A_n is generated by the 3-cycles $(1, 2, i)$, $2 < i \leq 3$.

(c) Let H be a subgroup of a group G of index n. Show that G has a normal subgroup N which is contained in H and which has index $\leq n!$.

3. (a) Let V be an noetherian module for a ring R, so that V satisfies the ascending chain condition on submodules. Let $T : V \to V$ be a surjective R-endomorphism. Prove that T is an isomorphism.

(b) In (a) suppose that R is a field, so V is a vector space. Give another explanation of (a) in terms of the rank and nullity of T.

4. (a) Find all the irreducible polynomials of degree 4 over the finite field $\mathbb{F}_2$.

(b) Let K/F be a finite extension of finite fields. Show the norm map $N_{K/F} : K \to F$ is surjective.

5. This problem tests some standard linear algebra facts. You can quote standard theorems.

Let F be a field and let $T : F^6 \to F^6$ be a linear operator with characteristic polynomial

$$\chi_T(t) = (t^2 + t + 1)(t^2 - 1)t^2.$$

(a) If $F = \mathbb{R}$, what are the various possibilities for the minimal polynomial $\mu_T(t)$ of T?

(b) Fill in the blank:

$$\det(T) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_; \qquad \mathrm{trace}(T) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Be careful about signs!

(c) Write down the companion matrix C of the polynomial $\chi_T(t)$. Calculate the minimal polynomial of C.

(d) When $F = \mathbb{C}$, when is T diagonalizable (i. e., represented by a diagonal matrix w.r.t. some basis)? Some explanation in terms of $\chi_T(t)$ or $\mu_T(t)$ is required.

(e) When $F = \mathbb{R}$, when (if ever) is T represented by a symmetric matrix? Why?

(f) Bonus (+4 points): Let $S : \mathbb{C}^m \to \mathbb{C}^m$ be a nilpotent operator which is represented by a matrix in Jordan normal form having blocks of sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Let $\lambda' = (\lambda'_1, \cdots, \lambda'_m)$ be the partition of m dual (or transpose) to the partition $\lambda = (\lambda_1, \cdots, \lambda_m)$ of m. What is the significance (in terms of T) of the integers λ'_i, $i = 1, \cdots, m$? (Note: your answer should be precisely one [short] sentence! No further explanation is wanted.)

6. (a) Suppose that $G = C_p \times \cdots C_p$ is a direct product of n copies of the cyclic group C_p of order p. How many subgroups does G have of order p? How many does it have of order p^{n-1}? Explain.

(b) Now let $p_1, \cdots, p_r$ be distinct prime integers > 0. Show that $F := \mathbb{Q}[\sqrt{p_1}, \cdots, \sqrt{p_r}]$ is an abelian Galois extension of $\mathbb{Q}$.

(c) Suppose that $a = p_{i_1} \cdots p_{i_m}$ (with distinct factors) is a nontrivial product of some of the primes $p_1, \cdots, p_r$. Let $b \neq a$ be another such element. Show that $\mathbb{Q}[\sqrt{a}] \neq \mathbb{Q}[\sqrt{b}]$. Now use (a) to determine precisely the Galois group of $F/\mathbb{Q}$. Carefully justify your answer.

(d) Show that the numbers $\sqrt{p_1}, \cdots, \sqrt{p_r}$ are linearly independent over $\mathbb{Q}$, and that $\sqrt{p_1} + \cdots + \sqrt{p_r}$ is a primitive element of $F/\mathbb{Q}$.

7. (a) Let G be a finite simple group of order 168. How many elements of order 7 does G have? Why?

(b) How many conjugacy classes of elements of order 7 does G have? Hint: By looking at Sylow 3-subgroups, show that G has no cyclic subgroup of order 21. Use this to determine the centralizer in G of an element of order 7. ....

(c) Assume that you know that $G := GL_3(\mathbb{F}_2)$ is a simple group. Explicitly exhibit two elements of G of order 7 which are not conjugate in G. Explain.

8. Let K be a field and let A, B be commutative K-algebras. We do not assume A or B is finite dimensional over K.

(a) If the K-algebra $A \otimes_K B$ is a field, show that A and B must be fields, too. (Partial credit is given if you have to assume that A or B is finite dimensional over K.)

(b) Provide an example of two field extensions A and B of degree 2 over K such that $A \otimes_K B$ is a field of degree 4 over K.

(c) Compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ explicitly.

(d) Suppose K has characteristic $p > 0$ and that A/K is a field extension such that there exists $\xi \in A$ such that $\xi \notin K$, but $\xi^p \in K$. Prove $A \otimes_K A$ is not a field.