Algebra general exam. August 20th 2010, 9am-2pm

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let p be prime. Let G be a finite group, K a normal subgroup of G, and assume that $|G/K|$ is divisible by p. Let P be a Sylow p-subgroup of G.

(a) (5 pts) Prove that PK/K is a Sylow p-subgroup of G/K

(b) (4 pts) Prove that $n_p(G/K)$ divides $n_p(G)$ where $n_p(\cdot)$ denotes the number of Sylow p-subgroups

(c) (3 pts) Prove that $n_p(G/K) = n_p(G)$ if and only if P is normal in PK.

2. Let F be a field, $M_2(F)$ the ring of 2×2 matrices over F and $GL_2(F)$ the group of invertible elements of $M_2(F)$.

(a) (4 pts) Prove that two matrices in $M_2(F)$ are similar if and only if they have the same minimal polynomial.

(b) (5 pts) Assume that F is finite, and let $q = |F|$. Find the number of conjugacy classes in $GL_2(F)$.

(c) (6 pts) Again assume that F is finite of order q. Find the number of nilpotent matrices in $M_2(F)$. **Hint:** If $A \in M_2(F)$ is nilpotent, what is its Jordan normal form? You may use without proof that $|GL_2(F)| = (q^2 - 1)(q^2 - q)$.

3. Let $K = \mathbb{Q}(\sqrt[3]{3}, \sqrt[5]{5})$, the field obtained from $\mathbb{Q}$ by adjoining $\sqrt[3]{3}$ and $\sqrt[5]{5}$.

(a) (3 pts) Prove that $[K : \mathbb{Q}] = 15$.

(b) (10 pts) Let $L \subseteq \mathbb{C}$ be the Galois closure of K over $\mathbb{Q}$, that is, L is the minimal Galois extension of $\mathbb{Q}$ which contains K. Describe L explcitly in the form $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_t)$, determine $[L : \mathbb{Q}]$ and describe the elements of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ by their actions on $\alpha_1, \alpha_2, \ldots, \alpha_t$.

(c) (5 pts) Prove that $K = \mathbb{Q}[\sqrt[3]{3} + \sqrt[5]{5}]$.

4. Let K be a field. Let A be a finite-dimensional (possibly non-commutative) K-algebra (with 1), and assume that A is a division ring.

(a) (5 pts) Prove that every K-subalgebra of A is a division ring. **Hint:** Let B be a K-subalgebra of A, take any $b \in B$, and consider the map $\mu_b : B \to B$ given by $\mu_b(x) = bx$.

(b) (5 pts) Assume that K is algebraically closed. Prove that $\dim_K A = 1$.

5. Let G be group. A subgroup H of G will be called *essential* if $H \cap K \neq \{1\}$ for every non-trivial subgroup K of G.

(a) (2 pts) Let p be a prime and $k \geq 2$. Prove that the group $\mathbb{Z}/p^k\mathbb{Z}$ has a proper essential subgroup.

(b) (7 pts) Assume that H_1 is an essential subgroup of G_1 and H_2 is an essential subgroup of G_2. Prove that $H_1 \times H_2$ is an essential subgroup of $G_1 \times G_2$.

(c) (6 pts) Let G be a finite abelian group. Prove that G does not have a proper essential subgroup if and only if G is a direct product of groups of prime order.

6. Let $\mathbb{R}$ denote the real numbers. The purpose of this problem is to show that the ring $A = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not a UFD. For an element $f \in \mathbb{R}[x, y]$ we denote its image in A by $[f]$.

(a) (2 pts) Show that every element of A can be uniquely represented in the form $[f(x) + g(x)y]$ where $f(x), g(x) \in F[x]$.

(b) (2 pts) Show that A has an automorphism φ of order 2 such that $\varphi([f(x)]) = [f(x)]$ for each $f(x) \in F[x]$ and $\varphi([y]) = -[y]$.

(c) (3 pts) Use (a) and (b) to construct a function $N : A \to F[x]$ such that $N(uv) = N(u)N(v)$ for all $u, v \in A$.

(d) (6 pts) Use the function N from (c) to show that $[x]$ is an irreducible element of A and that the only invertible elements of A are (images of) nonzero constant polynomials. **Hint:** It is essential that you are working over $\mathbb{R}$, not over $\mathbb{C}$.

(e) (4 pts) Now show that A is not a UFD.

7. Recall that a ring S is called *graded* if $S = \oplus_{n=0}^{\infty} S_n$ where each S_n is an additive subgroup and $S_n \cdot S_m \subseteq S_{n+m}$ for all n, m. An element $s \in S$ is called *homogeneous* if $s \in S_n$ for some n. An ideal I of S is called a *graded ideal* if $I = \oplus_{n=0}^{\infty} I \cap S_n$.

(a) (5 pts) Let S be a graded ring and I an ideal of S. Prove that I is a graded ideal if and only if I is generated (as an ideal) by a set of homogeneous elements.

(b) (8 pts) Let R be a commutative ring with 1 and $S = R[x]$. Then S is naturally a graded ring where $S_n = \{rx^n : r \in R\}$. Assume that there exists $k \in \mathbb{N}$ such that every ideal of R can be generated by at most k elements.

Let I be a graded ideal of S. Prove that I can be written as

$$I = J \oplus M$$

where J is an ideal of S generated by at most k elements and M is a finitely generated R-module. **Hint:** Adapt the proof of Hilbert's basis theorem.