Algebra general exam. August 17th 2009, 9am-1pm

Directions.

• Please show all your work and justify any statements that you make.
• State clearly and fully any theorem you use.
• Vague statements and hand-waving arguments will not be appreciated.
• You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let $G$ be a group of order 56 which does NOT have a normal subgroup of order 8.
   (a) (8 pts) Prove that $G$ has a normal subgroup of order 7.
   (b) (4 pts) Prove that $G$ has a subgroup of order 14.
   (c) (4 pts) Prove that $G$ has a normal subgroup of order 14.
   **Remark:** Of course, you may omit (b) if you correctly answered (c).

2. Let $G$ be a finite group and let $H$ and $K$ be subgroups of $G$. For each $x \in G$ define $HxK = \{hxk : h \in H, k \in K\}$.
   (a) (3 pts) Prove that for any $x, y \in G$ either $HxK = HyK$ or $HxK \cap HyK = \emptyset$.
   (b) (8 pts) Prove that $|HxK| = \frac{|H||K|}{|H \cap xKx^{-1}|}$. **Hint:** Use group actions: either a suitable action of $H \times K$ on $G$ or a suitable action of $H$ on $G/K$.

3. (a) (6 pts) Let $R$ be a principal ideal domain and $I \subset R$ a proper nonzero ideal. Prove that if the quotient ring $R/I$ is a domain, then it must be a field.
   (b) (4 pts) Does the assertion of (a) remain true if $R$ is only assumed to be a unique factorization domain? Prove or give a counterexample.

4. Let $F$ be a field and $R = F[x, y]$ the ring of polynomials over $R$ in two (commuting) variables $x$ and $y$. Let $I = xR$ be the principal ideal of $R$ generated by $x$ and $S = F + I = \{f + i : f \in F, i \in I\}$. Observe that $S$ is a subring of $R$ and $I$ is an ideal of $S$ (you need not justify these facts).
(a) (7 pts) Prove that \( I \) is not finitely generated as an ideal of \( S \).
**Hint:** Assume that \( I \) is finitely generated as an ideal of \( S \) and reach a contradiction by showing that there must exist a natural number \( m \) such that any polynomial \( p(x, y) \in I \) contains no monomials of the form \( xy^n \), with \( n > m \).

(b) (5 pts) Prove that \( S \) is not finitely generated as a ring.
**Hint:** It is possible to answer (b) using (a) without doing any computations.

5. (a) (8 pts) Let \( A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Mat}_3(\mathbb{C}) \). Find the minimal polynomial, the characteristic polynomial and the Jordan canonical form of \( A \).

(b) (7 pts) Let \( J_n(0) \in \text{Mat}_n(\mathbb{C}) \) be the Jordan block of size \( n \) with 0’s on the diagonal. Prove that there exists no matrix \( A \in \text{Mat}_n(\mathbb{C}) \) such that \( A^2 = J_n(0) \).

6. Let \( K/F \) be a finite extension of fields, and let \( \alpha, \beta \in K \) be such that \( K = F(\alpha, \beta) \). Let \( n = [F(\alpha) : F] \) and \( m = [F(\beta) : F] \), and assume that \( n \) and \( m \) are relatively prime.
   (a) (4 pts) Prove that \( [K : F] = nm \).
   (b) (6 pts) Assume that \( K/F \) is Galois. Let \( \mu_{\alpha,F}(x) \) and \( \mu_{\beta,F}(x) \) be the minimal polynomials of \( \alpha \) and \( \beta \) over \( F \), respectively. Let \( \alpha' \in K \) be a root of \( \mu_{\alpha,F}(x) \) and let \( \beta' \in K \) be a root of \( \mu_{\beta,F}(x) \). Prove that there exists unique \( \sigma \in \text{Gal}(K/F) \) such that \( \sigma(\alpha) = \alpha' \) and \( \sigma(\beta) = \beta' \).
   (c) (6 pts) Again assume that \( K/F \) is Galois. Let \( S \) be the set of all elements \( c \in F \) such that \( F(\alpha + c\beta) \neq K \). Prove that \( |S| \leq nm \).

7. Let \( F \) be a field and \( K \) a finite-dimensional vector space over \( F \). Let \( n = \dim_F K \), and assume that \( n > 1 \).
   (a) (6 pts) Is it always true that \( K \otimes_F K \cong \text{Mat}_n(F) \) as \( F \)-modules?
   (b) (6 pts) Now assume that \( K \) also has the structure of a commutative ring with 1, so being an \( F \)-vector space, \( K \) becomes an \( F \)-algebra. Recall that in this case \( K \otimes_F K \) possesses unique \( F \)-algebra structure such that \((a \otimes b) \cdot (c \otimes d) = ac \otimes bd \) for \( a, b, c, d \in K \). Prove that \( K \otimes_F K \) cannot be a field.
   **Hint:** Construct a non-trivial \( F \)-algebra homomorphism \( K \otimes_F K \to K \).

8. (8 pts) Let \( F \) be a field. Prove that the additive and multiplicative groups of \( F \) cannot be isomorphic. **Hint:** Look at the orders of elements in both groups.