Algebra general exam. August 17th 2009, 9am-1pm

Directions.

- Please show all your work and justify any statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

DO EACH PROBLEM ON A SEPARATE SHEET OF PAPER, AND STAPLE THEM TOGETHER IN THE CORRECT ORDER BEFORE TURNING THE EXAM IN.

1. Let G be a group of order 56 which does NOT have a normal subgroup of order 8.
(a) (8 pts) Prove that G has a normal subgroup of order 7.
(b) (4 pts) Prove that G has a subgroup of order 14.
(c) (4 pts) Prove that G has a normal subgroup of order 14.
Remark: Of course, you may omit (b) if you correctly answered (c).

2. Let G be a finite group and let H and K be subgroups of G. For each $x \in G$ define $HxK = \{hxk : h \in H, k \in K\}$.
(a) (3 pts) Prove that for any $x, y \in G$ either $HxK = HyK$ or $HxK \cap HyK = \emptyset$.
(b) (8 pts) Prove that $|HxK| = \dfrac{|H||K|}{|H \cap xKx^{-1}|}$. **Hint:** Use group actions: either a suitable action of $H \times K$ on G or a suitable action of H on G/K.

3. (a) (6 pts) Let R be a principal ideal domain and $I \subset R$ a proper nonzero ideal. Prove that if the quotient ring R/I is a domain, then it must be a field.
(b) (4 pts) Does the assertion of (a) remain true if R is only assumed to be a unique factorization domain? Prove or a give a counterexample.

4. Let F be a field and $R = F[x, y]$ the ring of polynomials over R in two (commuting) variables x and y. Let $I = xR$ be the principal ideal of R generated by x and $S = F + I = \{f + i : f \in F, i \in I\}$. Observe that S is a subring of R and I is an ideal of S (you need not justify these facts).

(a) (7 pts) Prove that I is not finitely generated as an ideal of S.
Hint: Assume that I is finitely generated as an ideal of S and reach a contradiction by showing that there must exist a natural number m such that any polynomial $p(x,y) \in I$ contains no monomials of the form xy^n, with $n > m$.

(b) (5 pts) Prove that S is not finitely generated as a ring.
Hint: It is possible to answer (b) using (a) without doing any computations.

5. (a) (8 pts) Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \in Mat_3(\mathbb{C})$. Find the minimal polynomial, the characteristic polynomial and the Jordan canonical form of A.
(b) (7 pts) Let $J_n(0) \in Mat_n(\mathbb{C})$ be the Jordan block of size n with 0's on the diagonal. Prove that there exists no matrix $A \in Mat_n(\mathbb{C})$ such that $A^2 = J_n(0)$.

6. Let K/F be a finite extension of fields, and let $\alpha, \beta \in K$ be such that $K = F(\alpha, \beta)$. Let $n = [F(\alpha) : F]$ and $m = [F(\beta) : F]$, and assume that n and m are relatively prime.
(a) (4 pts) Prove that $[K : F] = nm$.
(b) (6 pts) Assume that K/F is Galois. Let $\mu_{\alpha,F}(x)$ and $\mu_{\beta,F}(x)$ be the minimal polynomials of α and β over F, respectively. Let $\alpha' \in K$ be a root of $\mu_{\alpha,F}(x)$ and let $\beta' \in K$ be a root of $\mu_{\beta,F}(x)$. Prove that there exists unique $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\alpha) = \alpha'$ and $\sigma(\beta) = \beta'$.
(c) (6 pts) Again assume that K/F is Galois. Let S be the set of all elemetns $c \in F$ such that $F(\alpha + c\beta) \neq K$. Prove that $|S| \leq nm$.

7. Let F be a field and K a finite-dimensional vector space over F. Let $n = \dim_F K$, and assume that $n > 1$.
(a) (6 pts) Is it always true that $K \otimes_F K \cong Mat_n(F)$ as F-modules?
(b) (6 pts) Now assume that K also has the structure of a commutative ring with 1, so being an F-vector space, K becomes an F-algebra. Recall that in this case $K \otimes_F K$ possesses unique F-algebra structure such that $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$ for $a, b, c, d \in K$. Prove that $K \otimes_F K$ cannot be a field.
Hint: Construct a non-trivial F-algebra homomorphism $K \otimes_F K \to K$.

8. (8 pts) Let F be a field. Prove that the additive and multiplicative groups of F cannot be isomorphic. **Hint:** Look at the orders of elements in both groups.