General Exam January 9, 2008

Justify all your answers fully with a complete explanation on a *separate sheet of paper for each problem*, but put numerical answers in the boxes on the exam sheet. *Keep the problems in correct order* when you turn in your answers.

(1) (a) Show that a group G will have outer automorphisms (automorphisms which are not inner) if it can be properly imbedded as a normal subgroup $G \lhd G'$ of a group in such a way that $G \cdot \text{Centralizer}_{G'}(G) \neq G'$.
(b) Show that A_n has outer automorphisms whenever $n \geq 4$.
(c) Explain the mantra "Every automorphism of G is inner, somewhere."

(2) (a) Let R be a commutative ring with 1, and M a finite direct sum $M = M_1 \oplus \cdots \oplus M_n$ of simple unital left R-modules M_i. Show that M has both the ascending and descending chain conditions on R-submodules. (b) Where does your argument use the hypotheses that R is unital, R is commutative, or M is unital?

(3) Let V be an n-dimensional vector space V over a finite field F of q elements.
(a) Show that the number of invertible linear operators on V is $\prod_{i=0}^{n-1}(q^n - q^i)$.
(b) Find the cardinality $|P|$ of any 3-Sylow subgroup $P \leq G = \mathrm{GL}(4, \mathbb{F}_{81})$ where $n = 4$ and $F = \mathbb{F}_{81}$ is the field of $q = 81$ elements.

$|P| =$ [ ]

(c) Find the cardinality $|P'|$ of any 3-Sylow subgroup $P' \leq G'$ of the special linear group $G' = \mathrm{SL}(4, \mathbb{F}_{81})$ (those invertible 4×4 matrices of determinant 1) over a field $\mathbb{F}_{81}$ elements.

$|P'| =$ [ ]

(d) Describe up to isomorphism (in terms of Jordan canonical forms) all possible 3-torsion elements T of the general linear group $\mathrm{GL}(4, \mathbb{F}_{81})$: all invertible operators T with $T^{3^e} = Id$ for some $e \geq 0$. For each T list its minimum polynomial $\mu_T(x)$ and its 3-period (the smallest $e \geq 0$ with $T^{3^e} = Id$). [Hint: all eigenvalues of T already lie in $\mathbb{F}_3$.]

(4) (a) If $r \in \mathbb{Q}$ is a rational root of a monic integral polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x],$$

show that $r \in \mathbb{Z}$ is integral.
(b) Factor $y^5x + y^3x^2 + y + x^3$ into irreducible factors in $\mathbb{Z}[x, y]$, explaining why each is irreducible.

(5) Let R be a unital commutative ring, and consider 5 possible properties such a ring might have: it is ($\mathcal{P}_1$) a domain, ($\mathcal{P}_2$) a PID, ($\mathcal{P}_3$) Euclidean, ($\mathcal{P}_4$) noetherian, ($\mathcal{P}_5$) a UFD.

(a) For which n is it true that $R[x]$ *always* inherits property ($\mathcal{P}_n$) from R (if R has ($\mathcal{P}_n$), so *must* $R[x]$)? Circle the $n's$ for which this holds, and explain your answer or give a counterexample.

$n =$ | 1 2 3 4 5 |

(c) For which n is it true that R inherits property ($\mathcal{P}_n$) from $R[x]$? Circle the $n's$ for which this holds, and explain your answer or give a counterexample.

$n =$ | 1 2 3 4 5 |

(d) For which $n \neq 5$ is it true that a quotient R/P of R by a proper prime ideal ($P \triangleleft R, P \neq R$) inherits property ($\mathcal{P}_n$) from R? Circle the $n's$ for which this holds, and explain your answer or give a counterexample.

$n =$ | 1 2 3 4 |

(6) Let $R = \mathbb{Z} + 2\mathbb{Z}[x] = \mathbb{Z}1 + \sum_{n=0}^{\infty} 2\mathbb{Z}x^n$. (a) Show that R is not a UFD by finding an irreducible element that is not prime. (b) Show that R is not noetherian by showing that the ideal $2\mathbb{Z}[x]$ is not finitely generated: $2\mathbb{Z}[x] \neq \sum_{i=1}^{n} Rf_i(x)$ for any $f_i(x) \in 2\mathbb{Z}[x]$.

(7) Let F be a field.

(a) Show that if $a \in E$ is an element of a field extension of F with $[F(a) : F] = 7$, then $F(a^3) = F(a)$.

(b) Show that any subgroup of order 8 of the multiplicative group $F^{\times}$ of the field F *must* be cyclic. Is this also true of subgroups of order 16?

(8) (a) Find a splitting field E of the polynomial $x^4 + 3x^3 + 4x^2 + 3x + 3$ over the rationals $F = \mathbb{Q}$, and find its degree $[E : F]$. [Hint: write $E = F(\alpha, \beta)$ for an easy pure imaginary α and a real β.]

(b) Find the Galois group $Gal(E/F)$ of the extension field (describe all the automorphisms by their actions on α, β).

(c) Diagram the lattice of subgroups of the Galois group and the corresponding lattice of sub-field-extensions of E/F.